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**Zoology.** — *The Ampullae of Millepora.* By H. BOSCHMA.

(Communicated at the meeting of December 18, 1948.)

In 1884 QUELCH described the ampullae of *Millepora murrayi* Quelch as "receptacles covered by a very thin and porous layer, which is often broken away" (l.c., p. 539). In his report on the reef corals of the Challenger Expedition he described them as follows: "Ampullae developing as special cavities in the superficial meshwork of the coenenchyma; often crowded, about 0.75 mm. in diameter within, scarcely raised above the general surface, on which they are seen as small white spots or vesicles, which are about 0.5 mm. in diameter, the centre being generally pierced by a small pore" (QUELCH, 1886, p. 192).

HICKSON (1891 *a*) reported upon medusae found in the ampullae of *Millepora murrayi*, and in another paper (HICKSON, 1891 *b*) gave a detailed description of the male medusae present in the ampullae of this species. In a later paper HICKSON (1897) stated that he noticed several ampulla-bearing specimens other than those belonging to the species *Millepora murrayi*; of these he mentioned *M. schrammi* Duch. and Mich., *M. alcicornis* L., and *M. complanata* Lamk. In his next paper dealing with *Millepora* HICKSON (1898 *a*) remarked that ampullae may be found in plicate, ramose, and digitate specimens, and that the presence or absence of ampullae cannot be used as a specific character. In the same year HICKSON (1898 *b*) found male medusae in a specimen of the facies *"complanata"* (undoubtedly a specimen of *M. platyphylla* Hempr. and Ehr.) of the same size as those found previously in *M. murrayi*. According to HICKSON (1899) a brief account of medusae of a West Indian form of *Millepora* was given by DUERDEN (1899). In the cited paper HICKSON published a detailed description of the structure of various stages of the female medusae of this West Indian *Millepora*. Here HICKSON moreover stated that the medusae of *Millepora "complanata"* mentioned in his 1898 *b* paper were identical in size and form with those of *M. murrayi*, so that "no specific distinction could be drawn between the two forms based on characters of the medusa before it is set free" (HICKSON, 1899, p. 4). Finally mention must be made of figure 68 in HICKSON (1924), representing a part of a colony of *Millepora* (probably *M. alcicornis*), the surface of which is profusely pitted with open ampullae.

During my previous investigations on *Millepora* (BOSCHMA, 1948 *a*, *b*) I found open ampullae in *M. alcicornis*, and ampullae still in possession of their covering in *M. murrayi*, *M. latifolia*, and *M. platyphylla*. I noted a difference in size of the ampullae in the various specimens, but concluded that as yet the available data were too scanty to allow of a conclusion concerning specific differences.

Since I wrote my previous papers dealing with *Millepora* a continued examination of numerous colonies of the genus resulted in ten specimens with closed ampullae, some of which possessed a few only among a multitude of open ampullae, whilst in others most of the ampullae were still in a closed condition. Although this material is still far from being sufficiently complete it seemed worth while to try whether the ampullae might furnish peculiarities to be regarded as characteristic of the species. Hickson (1897, p. 3) wrote: "The most important problems that have still to be solved are these: Are the medusae of the different species of *Millepora* alike, or do they present specific differences? Are the medusae confined to the male sex, or do medusae occur bearing the ova?" Soon afterwards the second question could be answered in the affirmative (Hickson, 1899), but the first question still remains open. Moreover the question whether the ampullae of the different species of *Millepora* are alike or are presenting specific differences remains still unanswered. The new data contained in the present paper do show that the ampullae of different specimens of *Millepora* among each other may present striking differences, but it remains uncertain whether in all cases these data can be used for specific distinction.

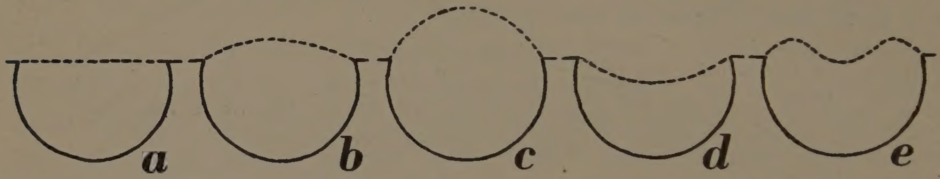

Fig. 1. Diagrams of sections of ampullae of various forms. Cavities of ampullae with full lines, coverings with broken lines.

For a comparison of the shape and size of the various ampullae it was necessary to obtain microphotographs of parts of the coralla, and I am strongly indebted to Professor S. T. Bok for placing his photographic apparatus at my disposal. As the surface of the corallum often is decidedly uneven in many cases it is difficult to obtain satisfactory results, but the photographs at least show the most important details of the ampullae.

In textfig. 1 diagrammatic sections of ampullae are represented showing the various forms of covering. Textfigs. 2 and 3 are more or less diagrammatic figures of some of the photographs of the plates, they may serve as an additional explanation of these photographs.

A short description of the ampullae of the various specimens follows here.

1. *Millepora alcicornis* L., Zoölogisch Museum Amsterdam (cf. Boschma, 1948 *b*, p. 101, no. 10). Locality unknown. Textfig. 4 *b* of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. I figs. 1—2, textfig. 2 *a*.

The colony shows a multitude of open ampullae, in certain parts only there are some ampullae which are still closed.

The ampullae have a diameter of 0.5—0.7 mm. They have a rather flat upper surface so that their shape approximately corresponds with the diagrammatic figure 1 *a*. The covering of the ampullae consists of trabeculae which have a strong tendency to a radial arrangement from the small central opening.

2. *Millepora alcicornis* L., Rijksmuseum van Natuurlijke Historie, Leiden (cf. BOSCHMA, 1948 *b*, p. 87, no. 3). Locality unknown. Textfig. 4 *a* of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. I figs. 3—5, textfig. 2 *c*.

The colony shows numerous ampullae, the greater part of which are still closed.

The ampullae have a diameter of 0.3—0.45 mm. Their central part is slightly sunk beneath the surface of the corallum, so that their shape more or less corresponds with the diagrammatic figure 1 *d*. The covering of the ampullae is a calcareous plate irregularly pierced by small openings surrounding the somewhat larger central opening.

3. *Millepora platyphylla* Hempr. and Ehr., Island Edam, Bay of Batavia, May 25, 1921. Colony of a plate-like growth.

Ampullae: Pl. II figs. 1—3, textfig. 2 *b*.

In certain regions the colony shows a large number of ampullae which for the greater part are still closed.

The ampullae have a diameter of 0.4—0.55 mm. Their covering as a rule is decidedly concave so that the central part is pronouncedly below the surface of the corallum (Pl. II figs. 1 and 3). In some parts of the colony the ampullae are less strongly concave (Pl. II fig. 2). Generally the shape of the ampullae corresponds with the diagrammatic figure 1 *d*, the concavity even may be still more pronounced. The covering of the ampullae consists of a calcareous plate which is more or less irregularly pierced by holes, the central of which usually is slightly larger than the others.

4. *Millepora platyphylla* Hempr. and Ehr., Island Edam, Bay of Batavia, date unknown. Colony of a plate-like growth (cf. BOSCHMA, 1948 *b*, Pl. XV fig. 5).

Ampullae: Pl. II figs. 4—7, textfig. 2 *d*.

In a part of the colony there are numerous ampullae, usually in a closed condition.

The ampullae have a diameter of 0.4—0.5 mm. They are slightly or pronouncedly concave, so that their general shape corresponds with the diagrammatic figure 1 *d*. The covering of the ampullae consists of a mass of trabeculae which are not distinctly arranged in a radial manner (Pl. II fig. 6), often the trabeculae are rather broadened, so that the covering consists of a calcareous plate pierced by irregular holes (Pl. II figs. 4, 5, 7). As a rule the central opening is somewhat larger than the other holes.

5. *Millepora murrayi* Quelch, British Museum (Natural History), reg. no. 94. 6. 19. 1, Tongatabu, J. J. LISTER. Colony of the same shape and

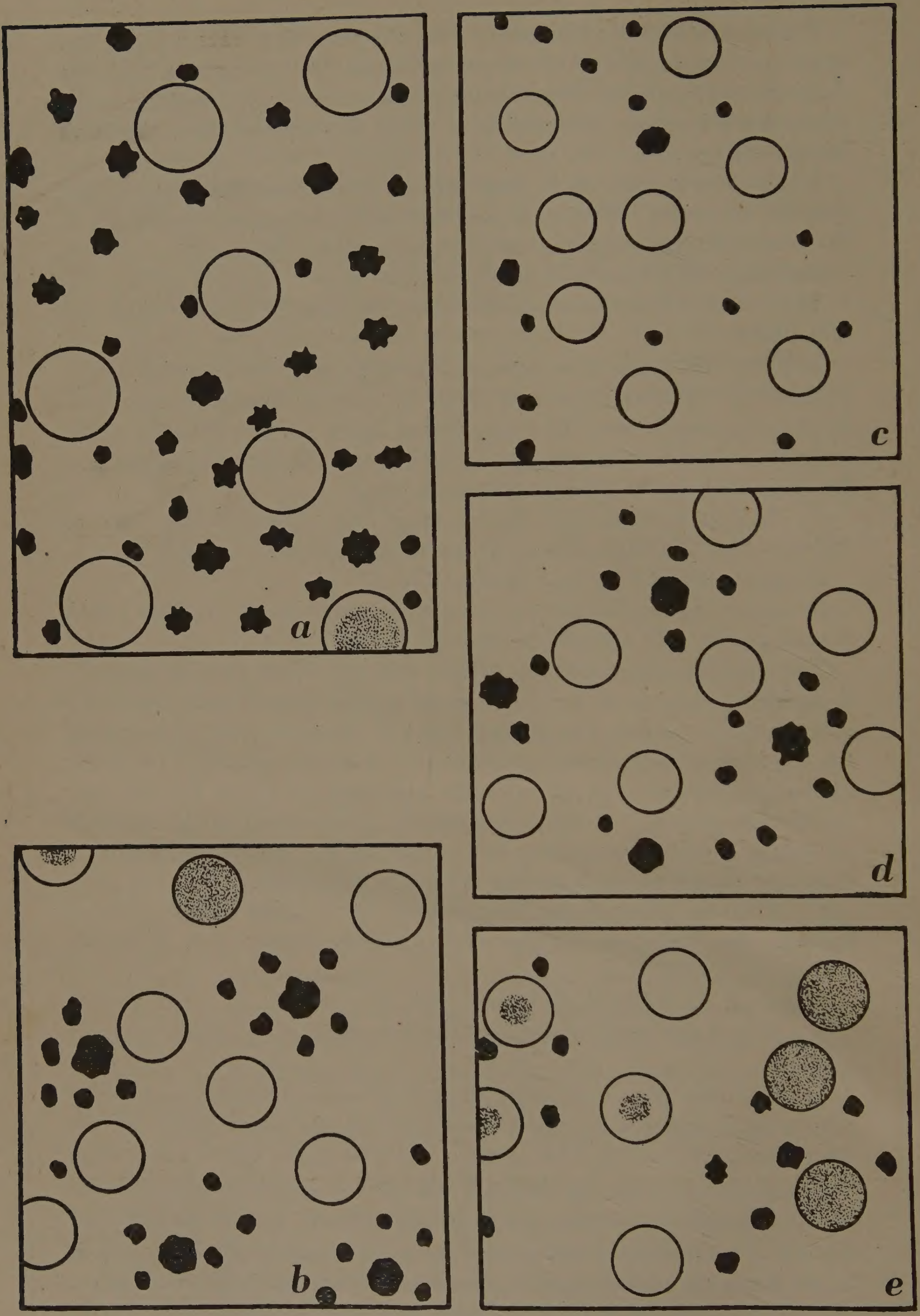

Fig. 2. Diagrams of some figures of the plates, indicating the place of the ampullae. *a, M. alcicornis* of Pl. I fig. 1; *b, M. platyphylla* of Pl. II fig. 2; *c, M. alcicornis* of Pl. I fig. 4; *d, M. platyphylla* of Pl. II fig. 7; *e, M. tenella* of Pl. V fig. 3. Open circles, closed ampullae; dotted parts, partly or wholly open ampullae; black, gastropores and dactylopores. × 20.

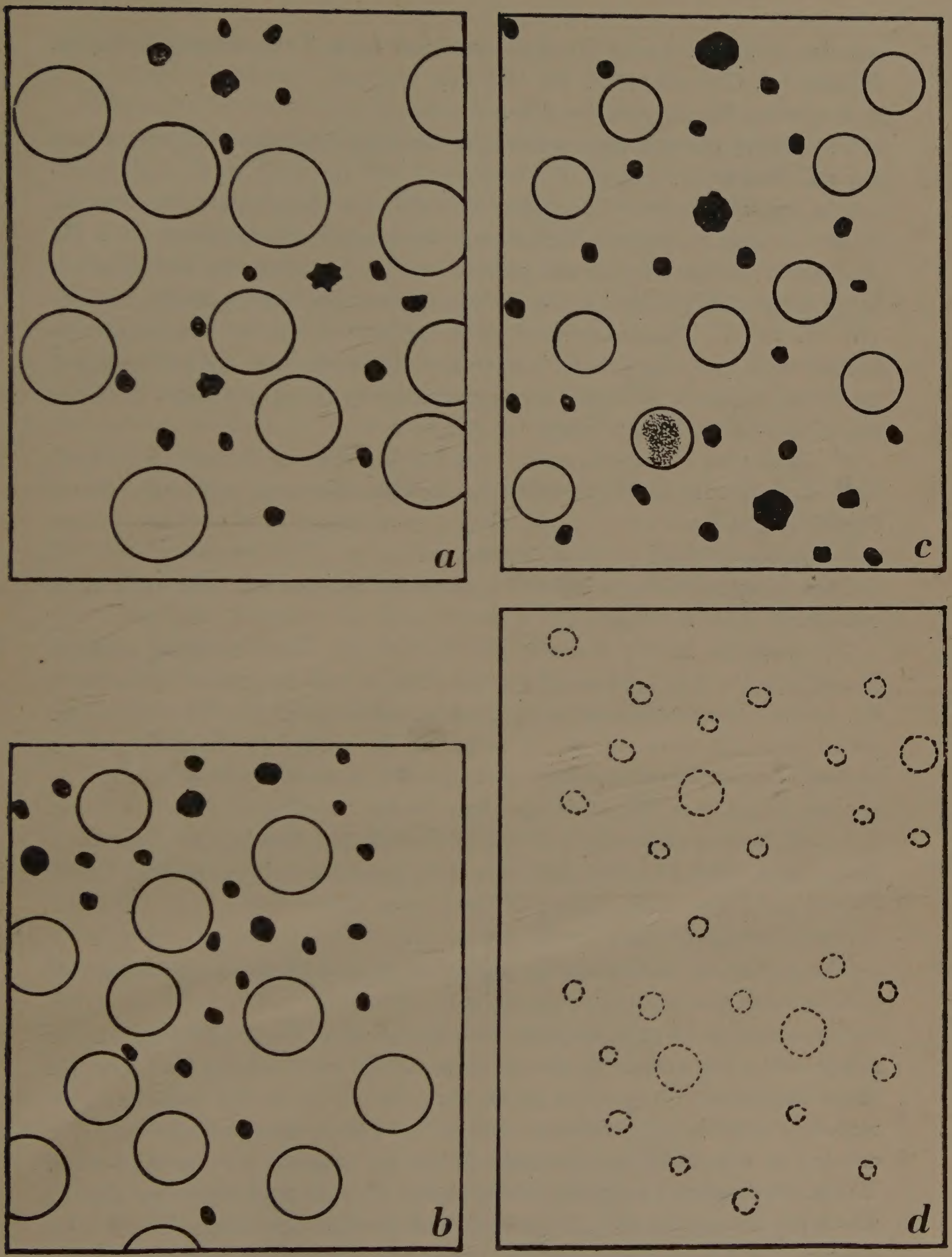

Fig. 3. Diagrams of some figures of the plates, indicating the place of the ampullae or (*d*) the partly closed pores. *a, M. murrayi* of Pl. III fig. 3; *b, M. murrayi* of Pl. III fig. 4; *c, M. tenella* of Pl. IV fig. 7; *d, M. platyphylla* of Pl. V fig. 3. For further explanation see fig. 2. × 20.

manner of branching as the type specimen from Zamboanga, Philippine Islands (cf. QUELCH, 1886, Pl. VII figs. 5—5 *a*).

Ampullae: Pl. III figs. 1—3, textfig. 3 *a*.

The colony shows a great number of ampullae which for the greater part are still closed.

The ampullae have a diameter of 0.4—0.7 mm. As a rule their covering is flat so that in general appearance the ampullae correspond with the diagrammatic figure 1 *a*. In the greater part of the colony the ampullae are fairly large (Pl. III figs. 1, 3), in other parts they are decidedly smaller (Pl. III fig. 2). The covering of the ampullae consists of a mass of trabeculae with comparatively wide openings. The trabeculae are not arranged in a radial manner. As a rule the central opening is slightly larger than the other openings of the covering.

6. *Millepora murrayi* Quelch, Island Edam, Bay of Batavia, July 5—8, 1921. Colony very similar to that described in a previous paper (BOSCHMA, 1948 *b*, Pl. XI fig. 2).

Ampullae: Pl. III figs. 4—6, textfig. 3 *b*.

The colony shows a multitude of ampullae, in closed as well as in open condition.

The ampullae have a diameter of 0.4—0.6 mm. Their covering projects noticeably above the surface of the corallum, so that in general appearance the ampullae correspond with the diagrammatic figure 1 *c*. The trabeculae which form the covering of the ampullae are rather thick, and between themselves they leave but small openings. The central opening as a rule is slightly larger than the other openings of the covering.

7. *Millepora tenella* Ortm., Snellius Expedition, Ake Selaka, Kaoe Bay, Halmahera, May 28, 1930. Rather widely branched colony, textfig. 4 *d* of the present paper is an outline of the topmost part.

Ampullae: Pl. IV figs. 1—4.

The surface of the colony is largely covered with ampullae, nearly all of which are still in a closed condition.

The ampullae have a diameter of 0.5—0.6 mm. Their marginal part is raised above the surface of the corallum, whilst their central part shows a slight depression. In general appearance the shape of the ampullae corresponds with the diagrammatic figure 1 *e*. The covering of the ampullae consists of a mass of largely fused trabeculae forming a calcareous plate. The latter is pierced by numerous openings of comparatively large size, of which not always the central opening is larger than the others. There is no distinct radial arrangement of the trabeculae.

8. *Millepora tenella* Ortm., British Museum (Natural History), reg. no. 76. 5. 5. 110, Rodriguez Island, SLATER. Colony of plate-like growth with short marginal branches, textfig. 4 *c* of the present paper is an outline of a fragment of this colony.

Ampullae: Pl. IV figs. 5—7, textfig. 3 *c*.

The colony shows a great number of ampullae, many of which still are in a closed condition.

The ampullae have a diameter of 0.3—0.5 mm. They have a flat surface or are slightly convex, so that their shape corresponds with the diagrammatic figure 1 *a* or *b*. The covering of the ampullae consists of a mass of rather thick trabeculae, leaving between themselves some small openings. There is no distinct radial arrangement of the trabeculae.

9. *Millepora tenella* Ortm., Siboga Expedition, station unknown. Rather delicate, widely spreading colony, textfig. 4 *e* of the present paper is an outline of some top branches of this colony.

Ampullae: Pl. V figs. 3—4, textfig. 2 *e*.

Many branches of the colony show a multitude of ampullae which for the greater part are in open condition; on a few branches there are some ampullae which have remained closed.

The ampullae have a diameter of 0.5—0.6 mm. They are slightly convex so that their general appearance corresponds with the diagrammatic figure 1 *b*. The covering of the ampullae consists of an irregular mass of thin trabeculae. The central opening is larger than the other openings between the trabeculae. There is no indication of a radial arrangement of the trabeculae.

10. *Millepora latifolia* Boschma, Island Edam, Bay of Batavia, May 27, 1921. Colony figured in a previous paper (BOSCHMA, 1948 *b*, Pl. IV fig. 1).

Ampullae: Pl. V figs. 1—2.

The colony shows a very great number of ampullae, in closed as well as in open condition.

The ampullae have a diameter of 0.6—0.8 mm. As a rule they are slightly convex (corresponding with the diagrammatic figure 1 *b*), some ampullae have a flat surface, and some are slightly concave. The covering consists of a system of trabeculae which more or less distinctly show an arrangement radiating from the centre. This holds at least for the peripheral region of the ampullae, the central part of the covering is a calcareous plate with small openings irregularly surrounding the somewhat larger central opening.

The data given above show that the ampullae in many cases possess peculiarities which undoubtedly are characteristic of the species.

It is a curious fact that the ampullae of *Millepora platyphylla* are of comparatively small size, whilst the gastropores in this species are decidedly larger than those of the other species of the genus. In one of the specimens of *M. platyphylla* the ampullae as a rule are pronouncedly concave (Pl. II figs. 1 and 3), though in the same colony there are regions in which the covering of the ampullae is hardly below the surface of the corallum (Pl. II fig. 2). Here they are of a quite similar appearance as the ampullae of the second specimen of *M. platyphylla* (Pl. II figs. 4—7). In the two specimens the covering of the ampullae consists of a similar mass of trabeculae in which a distinct radial arrangement cannot be observed. As far as definite conclusions may be drawn from the examination of the ampullae of the two colonies we may conclude that the ampullae of *M. platyphylla* differ from those of the other species of the genus by their strong tendency for a concave covering.

As compared with those of *M. platyphylla* the ampullae of *M. latifolia* (Pl. V figs. 1—2) have quite a different appearance. They are comparatively large (diameter 0.6—0.8 mm), they may be flat or slightly concave, but as a rule are more or less convex, and their covering consists of trabeculae which show a distinct tendency to a radial arrangement. In the two species the ampullae have such a pronouncedly different aspect that undoubtedly these differences point to specific distinction between *M. latifolia* and *M. platyphylla*.

Another specimen with ampullae of fairly large size is the colony of *M. murrayi* from Tongatabu. As a rule in this specimen the ampullae have a diameter of 0.5—0.7 mm (Pl. III figs. 1 and 3), in some parts of the colony they are somewhat smaller (0.4—0.5 mm, Pl. III fig. 2). The covering of the ampullae does not project above the surface of the corallum, it consists of a network of fine trabeculae in which a distinct radial arrangement cannot be observed. I could compare the ampullae of this specimen with those of the type specimen from the Philippine Islands, in the collection of the British Museum (Natural History). In the two specimens the ampullae are exactly alike in size and shape and structure, so that the ampullae of *M. murrayi* of Pl. III figs. 1—3 are of the typical shape. As far as concerns the two colonies taken as a whole the specimen from Tongatabu in every detail of shape and size of the branches and manner of branching corresponds with the type specimen from the Philippines.

In the specimen of *M. murrayi* from the Island Edam the ampullae have a diameter of 0.4—0.6 mm, and their covering consists of trabeculae which do not present a radial arrangement. They differ, however, from the other specimens examined by their pronouncedly convex shape of the covering (Pl. III figs. 4—6). This difference in shape of the ampullae might be an indication for a specific difference between the specimen from Tongatabu and that from the Island Edam. It must be admitted that it is not quite certain that the specimen from the Island Edam identified as *M. murrayi* really belongs to this species. The colony undoubtedly is a representative of the same species as the one figured in a previous paper (BOSCHMA, 1948 *b*, Pl. XI fig. 2), but this colony has not exactly the same manner of branching and shape and size of its component parts as typical specimens of *M. murrayi*. In the cited paper I came to the conclusion that there are ten distinct species of the genus *Millepora*. If this conclusion is right the specimen dealt with here undoubtedly belongs to *M. murrayi*. If on the other hand the two specimens of which the ampullae are shown on Pl. III are specifically distinct the specimen from the Island Edam is a representative of an undescribed species which closely resembles *M. murrayi*.

The differences in the ampullae of the two specimens, however, not necessarily point to a specific distinction. It remains possible that one of the colonies is of the male sex, the other of the female. It is unknown whether there are differences in shape and size between the medusae of either sex in one species of *Millepora* (male medusae only are known of two forms of Indopacific, and female medusae of one form of West Indian *Millepora*), but if these occur they might easily develop in ampullae of different shapes.

The ampullae of the two specimens of *M. alcicornis* again show striking differences. In the one colony (Pl. I figs. 1 and 2) the ampullae have a diameter of 0.5—0.7 mm, they have a rather flat surface and their covering shows a pronouncedly radial arrangement of trabeculae. In the other colony (Pl. I figs. 3—5) the ampullae are much smaller (diameter 0.3—

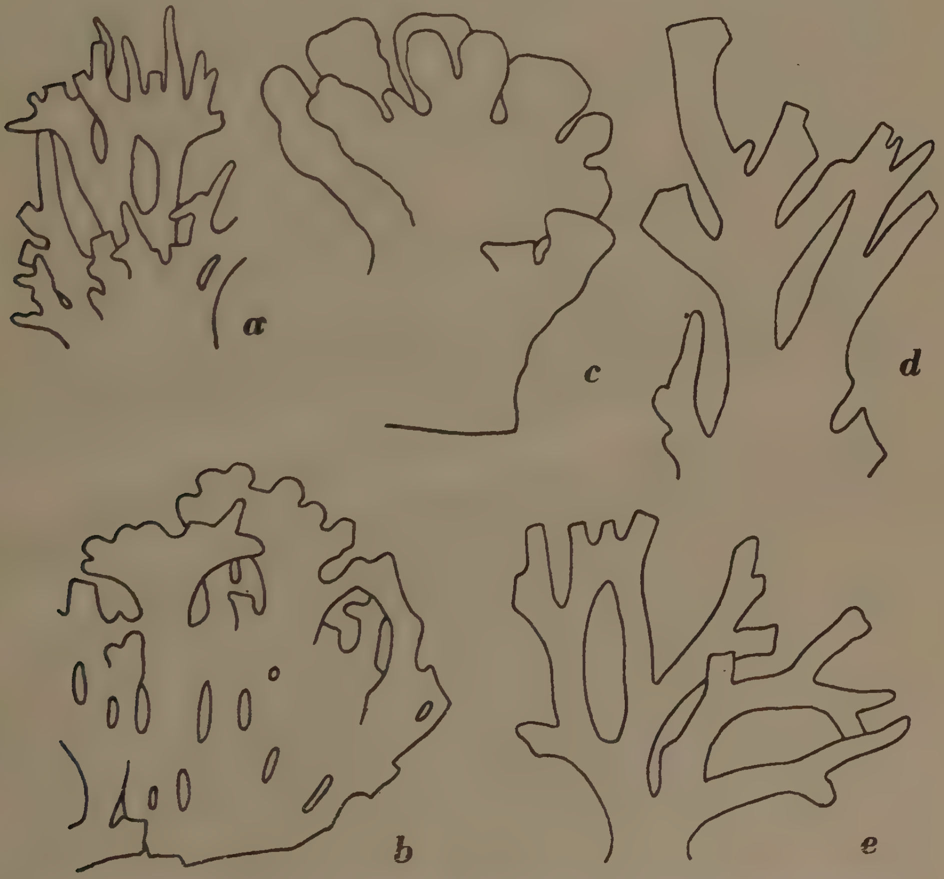

Fig. 4. Outlines of parts of colonies of *Millepora*. *a*, *M. alcicornis* from Leiden Museum; *b*, *M. alcicornis* from Amsterdam Museum; *c*, *M. tenella* from Rodriguez; *d*, *M. tenella* from Halmahera; *e*, *M. tenella* from Siboga Expedition. 5/6 natural size.

0.45 mm), they are slightly concave and their covering is composed of trabeculae which do not show a radial arrangement. These differences might point to a specific distinction of the two colonies, but it is highly improbable that we have to find a solution of the problem in this way. In their general appearance and in their manner of branching the two colonies are strongly similar. The fragment from which the photographs of Pl. I figs. 1 and 2 were taken is represented in outline in textfig. 4 *b*, a fragment of the colony from which the photographs of Pl. I figs. 3—5 were taken is shown in the same manner in textfig. 4 *a*. Both colonies are of the highly branched form of *Millepora alcicornis*, in both the branches have a strong tendency to unite into more or less plate-like forms. The differences in shape and in manner of branching of the two fragments are less striking than those often found in parts of the same colony.

Here the only possible explanation of the differences between the two sets of ampullae seems to be that one set represents the ampullae of the female colonies, the other set the ampullae of the male colonies.

When we compare the ampullae of the specimen of *M. tenella* from Rodriguez (Pl. IV figs. 5—7) to those of *M. tenella* from the Siboga Expedition (Pl. V figs. 3—4) we do not find striking differences. In the Rodriguez specimen the ampullae are rather small (diameter 0.3—0.5 mm), in the Siboga specimen they are slightly larger (diameter 0.5—0.6 mm), but in both specimens they are rather flat and do not show a pronouncedly radial arrangement of the trabeculae. The ampullae of the specimen of *M. tenella* from Halmahera have a diameter of 0.5—0.6 mm, they have no distinct radial arrangement of the trabeculae, but they differ in shape from those of the two other specimens as they do not possess a flat surface but have a convex marginal part and a concave central part. As a result they appear much more distinct than those of the two other specimens.

The difference in the shape of the ampullae in the three specimens of *M. tenella* remind of the differences found in the two specimens of *M. platyphylla*. In two specimens of *M. tenella* the ampullae have a more or less flat surface, in the third specimen they distinctly protrude over the surface of the corallum. These differences may be due to variation, possibly as a result of different circumstances of the localities in which the specimens were growing. On the other hand here again the differences in the shape of the ampullae may be the result of sexual differences of the colonies.

The three colonies of *M. tenella* in which closed ampullae were observed do not show important differences in shape and in manner of branching. The colony from Rodriguez (cf. textfig. 4 *c*) is of a more or less plate-like growth with short and blunt small branches in its marginal part, that from the Siboga Expedition (cf. textfig. 4 *e*) is of a slender growth form with spreading branches, whilst that from Halmahera (cf. textfig. 4 *d*) in its growth form is more of less intermediate between the two others, it is spreadingly branched with slightly more robust branches than the Siboga specimen. Among each other the three specimens do not present more striking differences than those commonly found in different parts of one fairly large colony of *M. tenella*.

It is highly probable that ampullae develop in the colonies of *Millepora* in certain times of the year only. Unfortunately only a small part of the material dealt with here has the dates of collecting. One specimen of *M. platyphylla* was collected on May 25, the specimen of *M. murrayi* from the Island Edam on July 5—8, the specimen of *M. tenella* from Halmahera on May 28, the specimen of *M. latifolia* on May 27, the specimen of *M. murrayi* from Zamboanga (Philippine Islands) between October 24 and November 12 or between January 11 and February 5 (cf. MOSELEY, 1879). The time of development of the ampullae may be different in the various species. The dates given above do not point to a distinct season for all the species of the genus.

The peculiarities of one more colony of *Millepora* need to be described here, as it presents trabecular plates which easily might be mistaken for the coverings of ampullae. The colony is from Station 193 of the Siboga Expedition (Sanana Bay, Soela Besi, September 13—14, 1899), it is the topmost part of a plate-like growth of *Millepora platyphylla* (breadth of the fragment 14 cm, height 12 cm) which has broken off from the colony and apparently was collected in living condition after having been lying for some months on one of the flat sides of the colony on the reef. A large part of the new lower surface of the colony now was almost completely shut off from light, and on this surface it is to be observed that the size of the pores is gradually diminishing as their openings become covered by a thin plate consisting of trabeculae growing inwards from the margin of the pores. The figures (Pl. V fig. 5 and textfig. 3*d*) show the surface of a part of this colony that remained in darkness for some time, here the gastropores and the dactylopores have become almost completely closed by the trabecular plates. Especially the almost closed gastropores show a marked resemblance to the ampullae of a number of specimens dealt with here.

Summarizing we may observe that the closed ampullae of the various specimens examined present striking differences which may be characteristic of the species. As long as it remains unknown whether or not the ampullae of a male colony of a certain species are different from those of a female colony of the same species no definite facts concerning the value of the ampullae as indicators of specific characters can be established.

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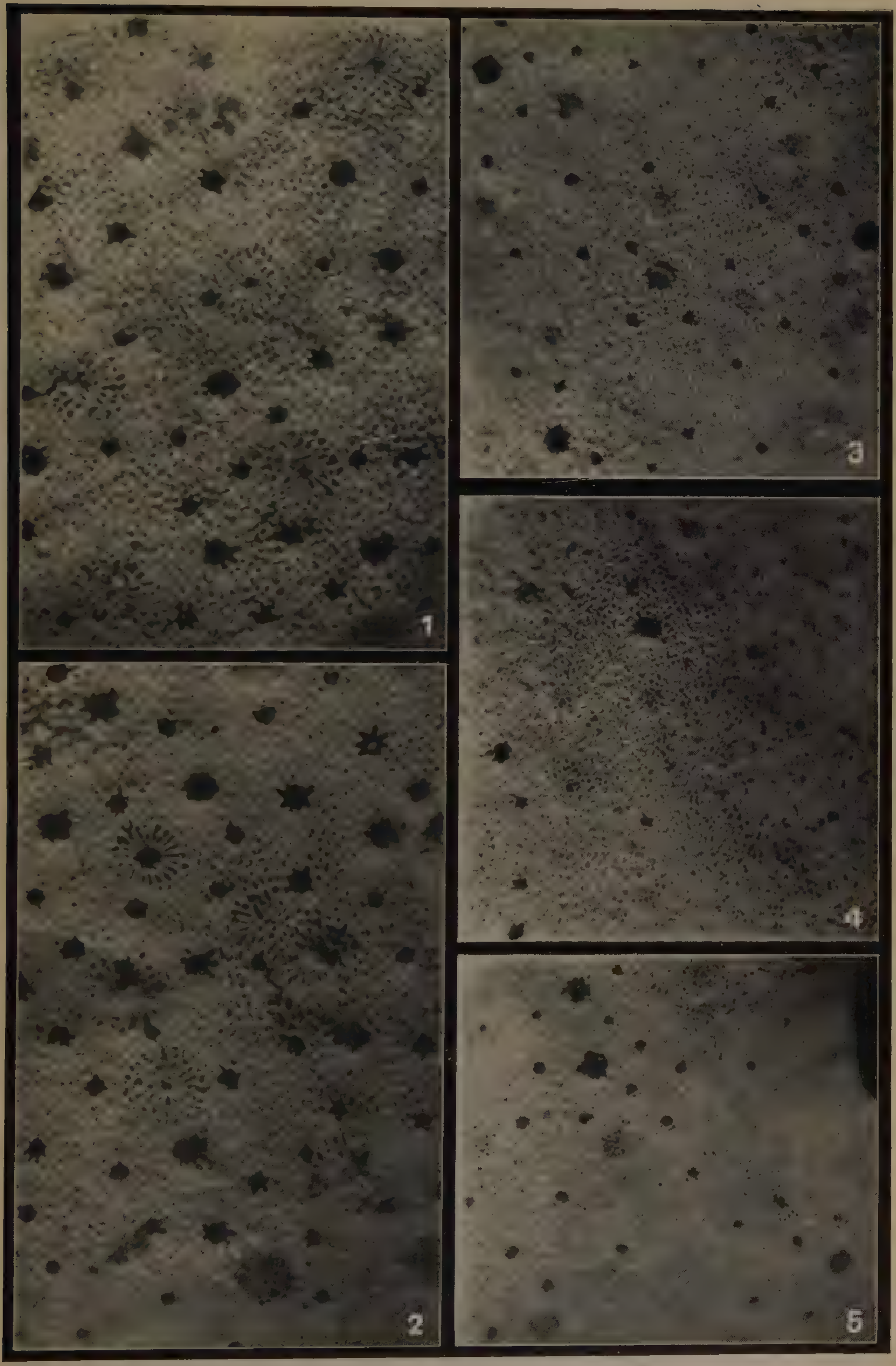
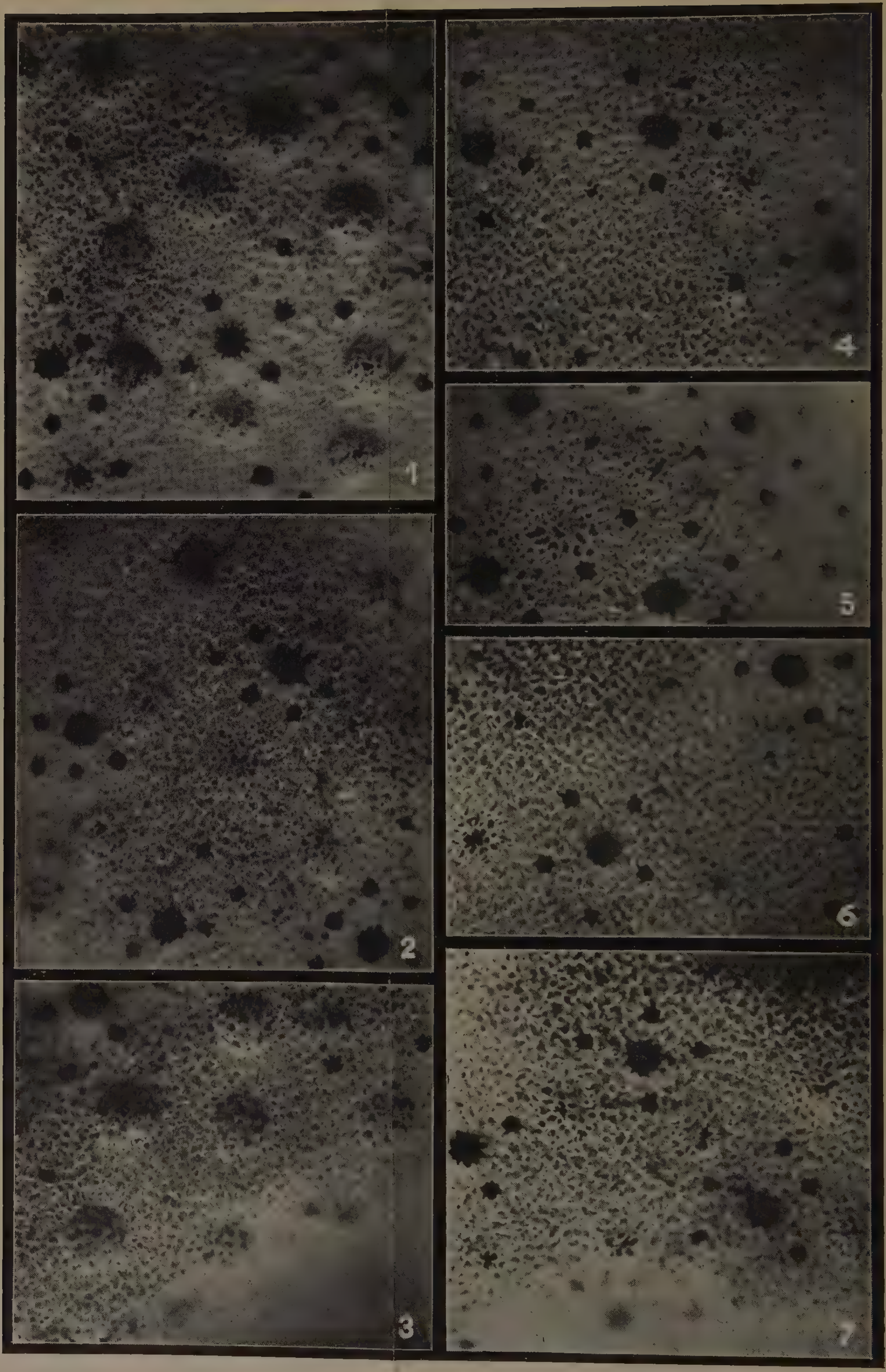
PLATE I

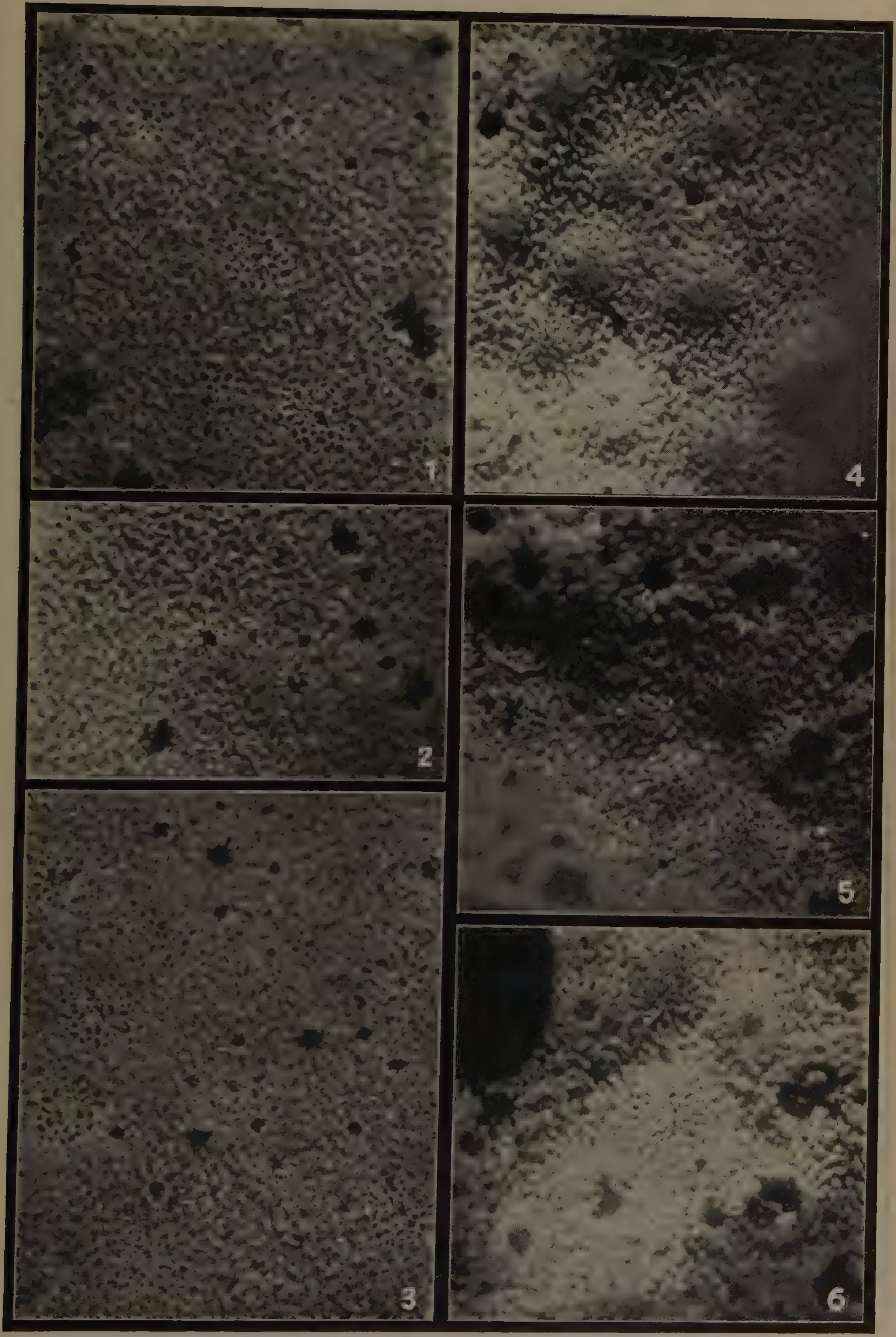


Figs. 1 and 2. *Millepora alcicornis* L., Amsterdam Museum, ampullae.
Figs. 3—5. *Millepora alcicornis* L., Leiden Museum, ampullae.
All figures × 20.



Figs. 1—3. *Millepora platyphylla* Hempr. & Ehr., Island Edam, May 25, 1921, ampullae.
Figs. 4—7. *Millepora platyphylla* Hempr. & Ehr., Island Edam, date unknown, ampullae.
All figures × 20.

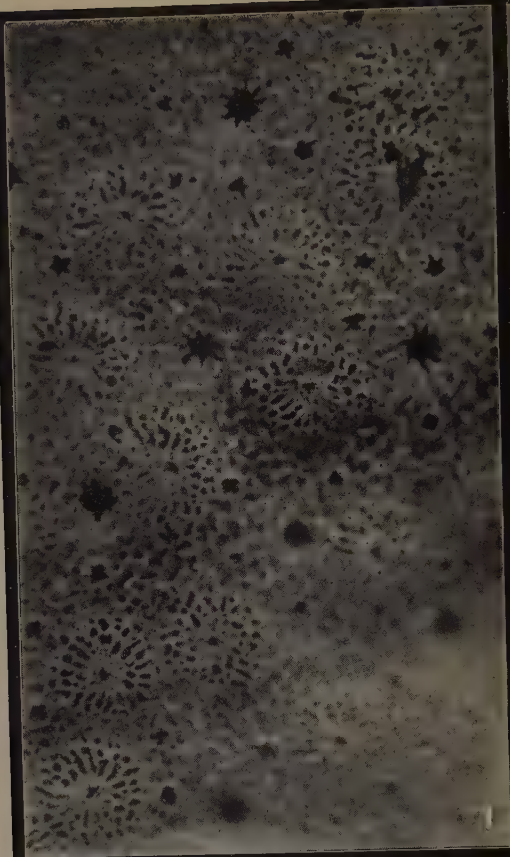
PLATE III

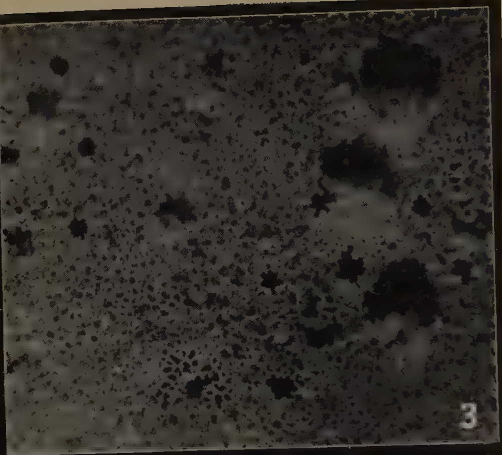


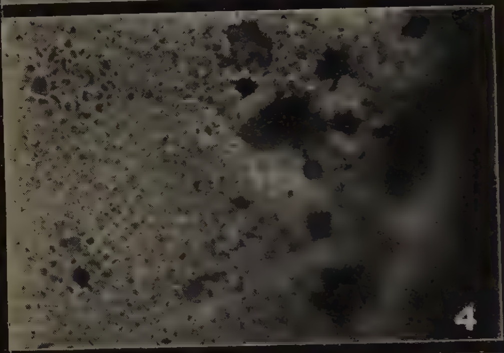
Figs. 1—3. *Millepora murrayi* Quelch, Tongatabu, ampullae.

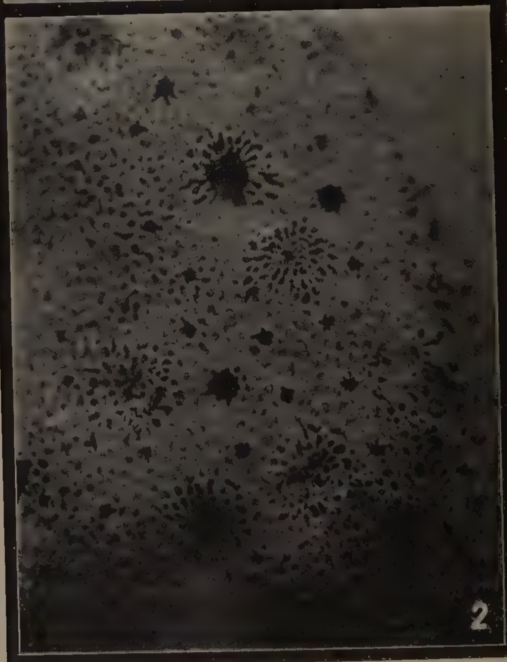
Figs. 4—6. *Millepora murrayi* Quelch, Island Edam, ampullae.

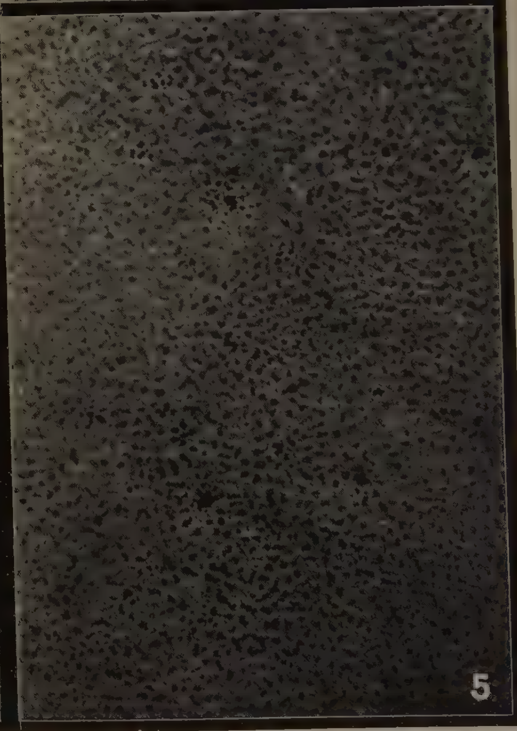
All figures × 20.











Figs. 1—2. *Millepora latifolia* Boschma, Island Edam, ampullae.
Figs. 3—4. *Millepora tenella* Ortm., Siboga Expedition, station unknown, ampullae.
Fig. 5. *Millepora platyphylla* Hempr. & Ehr., Siboga Expedition, Station 193, partially closed gastropores and dactylopores. All figures × 20.

**Biochemistry.** — *Elastic-viscous oleate systems containing KCl* [1]. II. a) *Period and logarithmic decrement as function of the radius of the sphere for a system containing 1,2 % oleate, in 1,52 N KCl + + 0,08 N KOH at 15° and 23° C;* b) *Shear modulus and relaxation time as function of the temperature.* By H. G. BUNGENBERG DE JONG and H. J. VAN DEN BERG.

(Communicated at the meeting of December 18, 1948.)

## 1. *Introduction.*

The object of the investigations described in the following communication was to obtain insight into the character of the damped oscillations of KCl containing oleate systems [2]. This was possible with the aid of formulae, developed by J. M. BURGERS [3] on the basis of older work by LAMB, giving the period $T$ and the logarithmic decrement $\Lambda$ for an elastic medium completely filling a spherical space, performing one of the three types of oscillations, already mentioned in Part I. BURGERS has considered three possible causes of damping:

(*a*) purely viscous damping; (*b*) damping through relaxation of elastic tensions, characterized by a single constant relaxation time; (*c*) damping through slipping along the wall of the vessel.

It is found in all cases that the period is directly proportional to the radius; hence the experimental confirmation of this proportionality (already obtained in Part I for the case of the rotational oscillation) does not allow to distinguish between the three possible causes of damping. Such a distinction, however, can be made when data are available concerning the dependence of the logarithmic decrement $\Lambda$ on the radius $R$, as in the case of viscous damping $\Lambda \sim R^{-1}$, in the case of relaxation damping $\Lambda \sim R$, while in the case of damping through slipping $\Lambda$ appears to be independent of $R$.

## 2. *Experimental method.*

As has been mentioned in preliminary investigations [4] on the elastic behaviour of KCl containing oleate systems, the elastic properties are a function of the KCl concentration, and it was found that in the case of the rotational oscillation the number of visible oscillations $n$ has a maximum for a definite value of the KCl content.

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[1] Part I has appeared in these Proceedings **51**, 1197 (1948).

[2] In the execution of the measurements we were assisted by D. VREUGDENHIL, to whom we express our thanks also here.

[3] J. M. BURGERS, these Proceedings **51**, 1211 (1948).

[4] H. G. BUNGENBERG DE JONG and G. W. H. M. VAN ALPHEN, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 1227 (1947).

For the present work we have used a concentration, nearly corresponding to this maximum of *n*, so that the damping was as small as possible. In order to prevent the appearance of auto-sensibilisation as a consequence of hydrolysis, it is necessary to ensure a sufficiently high pH (above 11), for which purpose a small amount of KOH is added [5]).

The solution used was prepared by mixing 4,5 litres of a solution of Na-oleate (90 gr Na-oleinicum pur. pulv. "Merck" [6]), dissolved in 4050 $cm^3$ aqua destillata to which is added afterwards 450 $cm^3$ KOH 2 N) with 3 litres of KCl 3,8 N. The mixture was thoroughly shaken for an extensive period in order to make it homogeneous, after which it was put into a thermostate at 15,0° C for two days, in order to obtain the desired temperature and to get rid of the air content. (In Part I we have already mentioned the disturbances that are caused by the presence of air bubbles).

The measurements were then started; the rotational, the meridional and the quadrantal oscillations were measured on three consecutive days; for details concerning the method applied we refer to Part I. We mention that the experiments with the very large reservoirs (the largest one has a capacity of 6 litres) had to be done outside the thermostate. The vessels were put on the turning table, described in Part I, and found themselves during 20 minutes in air of 18° C. The very large volume of these reservoirs and the low rate of heat exchange with the surroundings, combined with the relatively small values of the temperature coefficients of period and damping ratio for this system in the range of temperatures from 2° to 19° C (see Part I), make this permissible. It will appear that even in the less favourable case of measurements with a solution at a temperature of 23° C, the results concerning the functional relation between $T$ and $R$ or between $\Delta$ and $R$ for a single type of oscillation still were satisfactory.

It is possible nevertheless that results obtained in this way are less satisfactory when it is desired to make comparisons between the three types of oscillation. There are indications that such comparisons can safely be made only when data are obtained with a vessel completely surrounded by thermostate water.

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[5]) H. G. Bungenberg de Jong and G. W. H. M. van Alphen, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 849 (1947).

H. G. Bungenberg de Jong, H. L. Booij and G. G. P. Saubert, Protoplasma, **29**, 536 (1938).

[6]) The concentration of Na-oleate in the final mixture is 1,2% = 0,04 N Na-oleate. Relatively to the large concentration of K-ions (1,60 N) the small concentration of the Na-ions is negligible, so that we may consider our system practically as a K-oleate system.

In principle it might be possible to dissolve oleic acid in KOH, in order to arrive at a system containing exclusively K-ions; actually, however, this procedure proved to be unsatisfactory, which probably must be ascribed to the circumstance that the liquid oleic acid is much more liable to chemical alterations than the solid Na-oleate. Nevertheless also the latter substance gradually changes, even when it is left in its original packing (paraffined bottles, not opened). Pure Na-oleate when shaken must behave as a fine dusty powder. After deterioration it takes the form, first of coarse grains, finally of a compact viscous mass. Compare the first paper mentioned in footnote [5]).

### 3. *Period and logarithmic decrement as functions of R. Measurements at 15° C.*

Owing to the difficulties inherent in the measurement of the damping ratio, it was necessary to use vessels of large radius. For this purpose it was decided to use Pyrex and Jena "round bottom" vessels, of nominal capacities of 6 litres, 3 litres, 1,5 litres and 750 $cm^3$. The actual volumes contained in these vessels till the beginning of the tube-shaped neck were: 6000, 3309, 1738 and 754 $cm^3$ respectively. Assuming an exact spherical shape, the radii become: 11,27; 9,24; 7,46; 5,65 cm.

TABLE I.

Measurements with a 1,2 % oleate system (containing 1,52 N KCl + 0,08 N KOH) at 15.0° C.

| Type of oscillation | $R$ (cm) | $n$ | $10 \times \frac{T}{2}$ (sec) | $b_1/b_3$ | $\Lambda$ $\left(=2.303 \log \frac{b_1}{b_3}\right)$ | $10 \times \frac{T}{2}$ corr. | $\lambda$ (sec) $\left(=\frac{T}{2\Lambda}\right)$ | | G (dynes/$cm^2$) | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 11.27 | 28.8 ± 0.1 | 12.31 ± 0.020 | 1.702 ± 0.029 | 0.532 ± 0.017 | 12.27 | 2.31 | | 44.4 | |
| | 9.24 | 32.2 ± 0.2 | 10.09 ± 0.029 | 1.553 ± 0.026 | 0.440 ± 0.017 | 10.07 | 2.29 | | 44.3 | |
| otational | 7.46 | 37.5 ± 0.3 | 8.28 ± 0.024 | 1.408 ± 0.014 | 0.342 ± 0.010 | 8.27 | 2.42 | mean | 42.8 | mean |
| | 5.65 | 42.6 ± 0.2 | 6.41 ± 0.015 | 1.352 ± 0.008 | 0.301 ± 0.006 | 6.40 | 2.13 | 2.29 | 41.0 | 44.1 |
| | 2.33* | 47.5 ± 0.5 | 2.44 ± 0.015 | 1.111 ± 0.003 | 0.105 ± 0.003 | 2.44 | 2.32 | | 48.0 | |
| | 11.27 | 29.5 ± 0.3 | 9.39 ± 0.019 | 1.440 ± 0.014 | 0.365 ± 0.009 | 9.37 | 2.57 | | 46.2 | |
| eridional | 9.24 | 33.5 ± 0.2 | 7.67 ± 0.018 | 1.385 ± 0.023 | 0.326 ± 0.016 | 7.66 | 2.35 | mean | 46.5 | mean |
| | 7.46 | 34.0 ± 0.3 | 6.26 ± 0.015 | 1.267 ± 0.006 | 0.237 ± 0.004 | 6.26 | 2.64 | 2.52 | 45.4 | 45.9 |
| | 5.65 | 37.3 ± 0.3 | 4.74 ± 0.020 | 1.207 ± 0.004 | 0.188 ± 0.003 | 4.74 | 2.52 | | 45.4 | |
| | 11.27 | 35.1 ± 0.3 | 7.87 ± 0.024 | 1 369 ± 0.009 | 0.314 ± 0.007 | 7.86 | 2.50 | | 44.6 | |
| uadrantal | 9.24 | 40.0 ± 0.3 | 6.33 ± 0.010 | 1.310 ± 0.012 | 0.270 ± 0.009 | 6.32 | 2.34 | mean | 46.4 | mean |
| | 7.46 | 43.3 ± 0.3 | 5.13 ± 0.016 | 1.231 ± 0.006 | 0.208 ± 0.005 | 5.13 | 2.47 | 2.38 | 45.9 | 45.7 |
| | 5.65 | 47.2 ± 0.3 | 3.89 ± 0.013 | 1.193 ± 0.005 | 0.176 + 0.005 | 3.89 | 2.21 | | 45.8 | |

* In this case an approximately spherical reservoir of 52,7 $cm^3$ capacity has been used, immersed in the thermostate of 15° C. The damping ratio could be measured in the case of the rotational oscillation only, as for the other two forms the period of the oscillation was too short.

Table I for the temperature of 15° C gives the experimental results concerning: type of oscillation, total number $n$ of visible oscillations; $10 \times T/2$ (*i.e.* the time for 10 turning points, that is 5 complete oscillations); the damping ratio $b_1/b_3$ calculated from the position of 4 consecutive turning points, read off on the scale of the ocular micrometer [7]); and the corrected value of $10 \times T/2$, obtained with the aid of the formula

$$T_{\text{corr}} = \frac{T}{\sqrt{1+(\Lambda/2\pi)^2}}.$$

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[7]) The measurements of the period and of the damping ratio are performed with the aid of the telescope of a kathetometer, from a distance of approximately 1 m. The total

The last two columns of the table will be considered later. All data given are averages derived from 10 measurements of $n$, from 20 measurements of $10 \times T/2$, and from 20 measurements of $b_1/b_3$. The mean error [calculated from $\sqrt{\Sigma \Delta^2/n(n-1)}$] has been stated in each case.

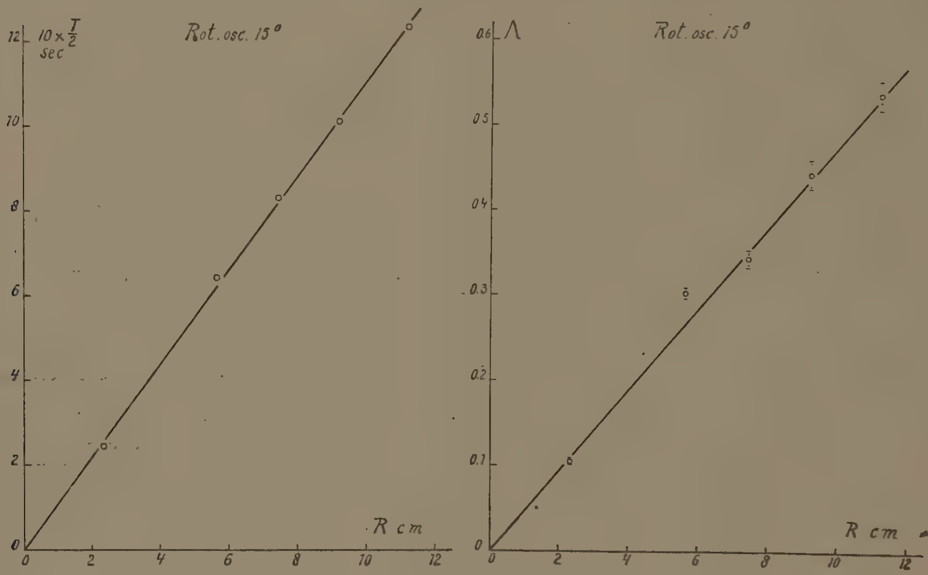

Fig. 1.

Fig. 1, 2 and 3 give the corrected values of $10 \times T/2$ and the values of $\Lambda$ in function of the radius $R$.

It is evident that the period is directly proportional to $R$, for each of the three types of oscillation. The points are situated so closely to a straight line passing through the origin, that we did not consider it necessary to indicate the mean errors in the diagrams. In the case of the decrement, where the spreading is larger, the errors have been represented. When due regard is given to these errors it is found that also the $\Lambda$-points, on the whole, are sufficiently close to a straight line through the origin. In two cases, however, the distance of the points from the line is much larger than the mean error. We must suppose that here some irregularity has occurred, the nature of which is unknown. Nevertheless the general conclusion is allowed that for the system considered both $T$ and $\Lambda$ are pro-

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length of the scale of the ocular micrometer (50 divisions) corresponds to a length of ca. 13 mm in the vessel filled with the oleate solution. The displacements of the air bubbles at first exceed the length of this scale, and measurements of the damping ratio can be carried out only when the amplitude has decreased. According to our notes the length $b_1$ (distance between the position of the bubbles at the first turning point and their position at the next turning point) ordinarily does not exceed 25 scale divisions, which represents an actual displacement of ca. 6 mm. The deviation from the equilibrium position thus is ca. 3 mm. For the points on which the telescope was focussed compare Part. 1.

portional to $R$, which proves that the damping in this case must be ascribed to relaxation with a constant relaxation time[8]).

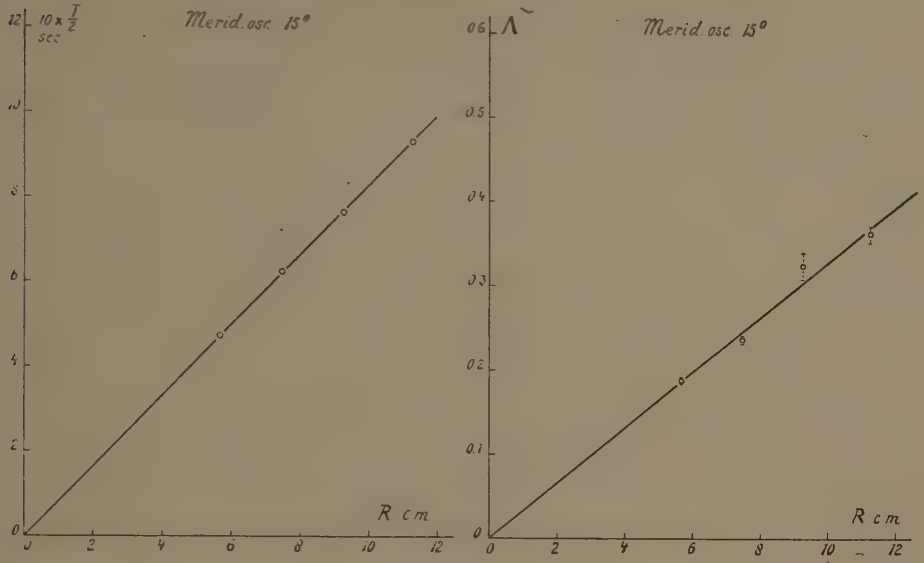

Fig. 2.

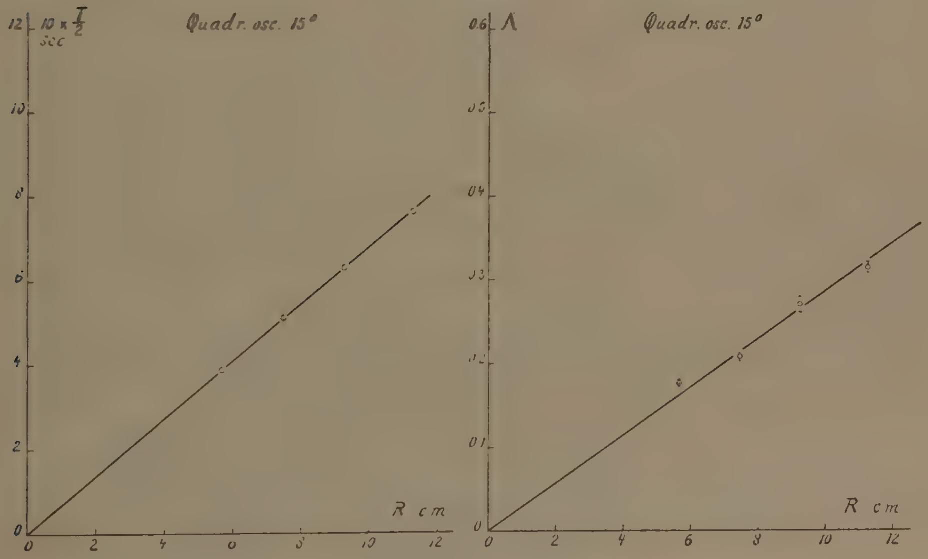

Fig. 3.

### 4. *Period and logarithmic decrement as functions of R. Measurements at 23° C.*

With the same oleate system similar measurements were performed at a temperature of 23° (two days later). Whereas the measurements described in the preceding section referred to a point on that branch of the $b_1/b_3$-temperature curve, which presents the smaller inclination, the higter temperature was chosen so as to give a point on the steep branch of this

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[8]) The reader must be warned against the idea that this result should apply to all elastic oleate systems. In the next communication we shall encounter a case where the relation between $\Lambda$ and $R$ is different.

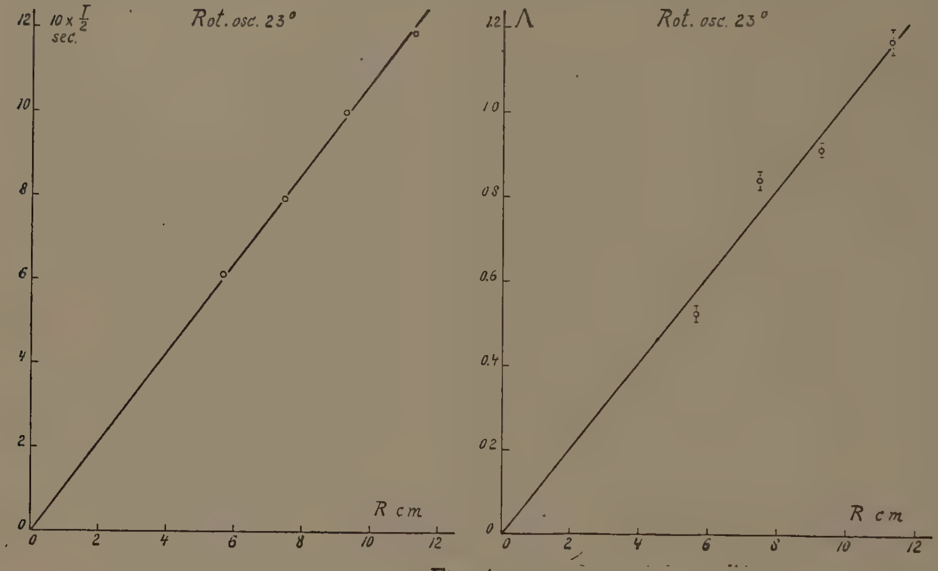

Fig. 4.

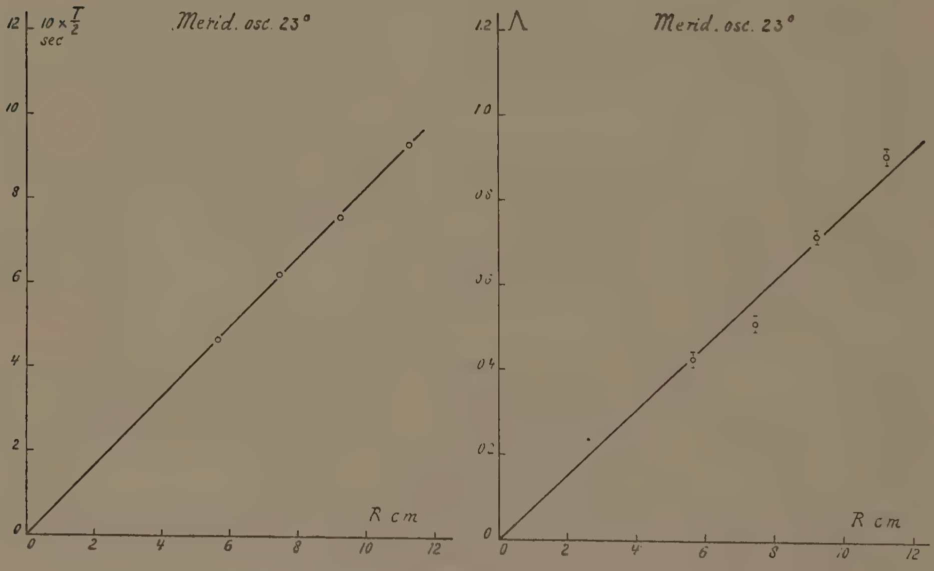

Fig. 5.

curve (compare Part I, fig. 4). The measurements have been executed in a single day, which, however, made it necessary to restrict to the rotational and the meridional types, as measurements with the quadrantal oscillation always are more difficult, in particular at the higher temperature.

The experimental results and the data derived from them are given in Table II.

TABLE II.

Measurements with the same system as that of Table I, at 23,0° C.

| Type of oscillation | $R$ (cm) | $n$ | $10 \times \frac{T}{2}$ (sec) | $b_1/b_3$ | $\Lambda$ $\left(= 2.303 \log \frac{b_1}{b_3}\right)$ | $10 \times \frac{T}{2}$ corr. | $\lambda$ (sec) $\left(= \frac{T}{2\Lambda}\right)$ | | $G$ (dynes/cm²) | |
|---|---|---|---|---|---|---|---|---|---|---|
| Rotational | 11.27 | 7.1 ± 0.1 | 12.04±0.058* | 3.195±0.098 | 1.162 ± 0.031 | 11.84 | 1.02 | mean 1.06 | 47.7 | mean 46.1 |
| | 9.24 | 9.3 ± 0.3 | 10.08±0.042* | 2.480±0.042 | 0.909 ± 0.017 | 9.98 | 1.10 | | 45.1 | |
| | 7.46 | 12.7 ± 0.2 | 8.00±0.027 | 2.311±0.048 | 0.838 ± 0.021 | 7.93 | 0.95 | | 46.6 | |
| | 5.65 | 17.0 ± 0.1 | 6.13±0.018 | 1.687±0.032 | 0.523 ± 0.019 | 6.11 | 1.17 | | 45.0 | |
| Meridional | 11.27 | 7.1 ± 0.1 | 9.38±0.060* | 2.473±0.055 | 0.906 ± 0.022 | 9.28 | 1.02 | mean 1.09 | 47.1 | mean 47.2 |
| | 9.24 | 10.1 ± 0.2 | 7.60±0.038* | 2.047±0.036 | 0.717 ± 0.018 | 7.55 | 1.05 | | 47.9 | |
| | 7.46 | 12.9 ± 0.1 | 6.21±0.017 | 1.669±0.032 | 0.512 ± 0.019 | 6.19 | 1.21 | | 46.4 | |
| | 5.65 | 17.1 ± 0.1 | 4.65±0.016 | 1.531±0.023 | 0.426 ± 0.017 | 4.64 | 1.09 | | 47.4 | |

* As the number of observable oscillations was too small, we have measured 5 × $T/2$ instead of 10 × $T/2$. The results and the mean errors all have been multiplied by 2.

It will be seen that the mean errors have become larger at the higher temperature, in particular those for the damping.

In figures 4 and 5 the corrected values of 10 × $T/2$ and the values of $\Lambda$ have been represented as functions of $R$.

The experimental points on the whole are again situated on straight lines through the origin, although in the case of $\Lambda$ the distances of the points from the line sometimes exceed the mean errors. We believe that it is permitted to consider these deviations as the results of unsystematic errors, and conclude that the relations $T \sim R$, $\Lambda \sim R$ are also valid at 23° C. Hence also at this temperature damping must be ascribed to relaxation with a constant relaxation time. We shall come back to this point in the last section of this paper.

## 5. *Coefficients occurring in* BURGERS' *formulae.*

BURGERS has given the following expressions for the period and the logarithmic decrement of an elastic medium, confined in a spherical space and describing oscillations damped through relaxation:

rotational oscillation ... $T_0 = \frac{2\pi}{4.49} R \sqrt{\frac{\varrho}{G}}$ and $\Lambda_0 = \frac{\pi}{4.49} R \frac{1}{\lambda} \sqrt{\frac{\varrho}{G}}$

meridional oscillation ... $T_1 = \frac{2\pi}{5.76} R \sqrt{\frac{\varrho}{G}}$ and $\Lambda_1 = \frac{\pi}{5.76} R \frac{1}{\lambda} \sqrt{\frac{\varrho}{G}}$

quadrantal oscillation ... $T_2 = \frac{2\pi}{6.99} R \sqrt{\frac{\varrho}{G}}$ and $\Lambda_2 = \frac{\pi}{6.99} R \frac{1}{\lambda} \sqrt{\frac{\varrho}{G}}$

in which:

$T$ = period for a complete oscillation in sec
$R$ = radius of the spherical vessel in cm
$\varrho$ = density of the elastic medium in gr/cm$^3$
$G$ = shear modulus in dyne/cm$^2$
$\lambda$ = relaxation time in sec.

When these formulae are applied to our oleate system they allow the calculation of the characteristic quantities $G$ and $\lambda$, so that it becomes possible to investigate the dependence of these quantities on various parameters (temperature, KCl concentration, oleate concentration etc.).

The formulae give a direct proportionality for $T$ and $\Lambda$ with $R$, which has already been confirmed experimentally.

The next point to be considered are the values of the ratios $T_0/T_1$, $\Lambda_0/\Lambda_1$, $T_0/T_2$, $\Lambda_0/\Lambda_2$, for which the equations give:

$$T_0/T_1 = \Lambda_0/\Lambda_1 = 5.76/4.49 = 1.283;$$
$$T_0/T_2 = \Lambda_0/\Lambda_2 = 6.99/4.49 = 1.557.$$

The values calculated from the experimental results (Table I) obtained at 15° C are given below:

| $R$ | $T_0/T_1$ | | $\Lambda_0/\Lambda_1$ | | $T_0/T_2$ | | $\Lambda_0/\Lambda_2$ | |
|---|---|---|---|---|---|---|---|---|
| 11.27 | 1.309 | | 1.458 | | 1.561 | | 1.694 | |
| 9.24 | 1.315 | mean | 1.350 | mean | 1.593 | mean | 1.630 | mean |
| 7.46 | 1.321 | **1.324** | 1.443 | **1.463** | 1.612 | **1.603** | 1.644 | **1.668** |
| 5.65 | 1.350 | | 1.601 | | 1.645 | | 1.704 | |

The mean values of the ratios found experimentally are higher than the theoretical ones: 3 % and 3 % respectively for the period, and 14 % and 7 % respectively for the decrement. It must be kept in mind, however, that, as our primary purpose was to find the relation between $T$ and $R$, and between $\Lambda$ and $R$ for a single type of oscillation, the three types have been investigated on consecutive days, so that perhaps the circumstances are not completely identical for the three types.

This supposition is supported by the following data, referring to the experiments (Table II) performed at 23° C:

| $R$ | $T_0/T_1$ | | $\Lambda_0/\Lambda_1$ | |
|---|---|---|---|---|
| 11.27 | 1.276 | | 1.283 | |
| 9.24 | 1.322 | mean | 1.268 | mean |
| 7.46 | 1.281 | **1.299** | 1.637 | **1.372 (1.260)** |
| 5.65 | 1.317 | | 1.228 | |

The measurements from which these numbers have been deduced were performed on a single day, and it will be seen that the difference between the experimental values of the ratios and the theoretical ones is less (1 %

for $T$, 14 % for $\Lambda$). Amongst the experimental values of $\Lambda$ there is one which apparently is in error (the value 1.637 for $R = 7.46$ cm); if this value is omitted, the mean becomes 1.260 (indicated between brackets), which deviates only 2 % from the theoretical value.

A further inspection of the data shows that the differences between the experimental and the theoretical values of the ratios generally increase with decrease of the radius of the vessel. It is possible that the deviation must partly be ascribed to the fact that the vessels, when put on the turning table, found themselves in air of a temperature differing from that inside the fluid. Whilst this circumstance was not of much importance in the investigations on the relation between $T$ and $R$ and that between $\Lambda$ and $R$, it has apparently been more harmful in the present case, which requires a higher degree of precision.

We therefore decided to perform new determinations with a single vessel, executing all measurements in one consecutive series, in such a way that heat exchange with the surroundings was excluded. For this purpose we put on the turning table a wide glass cylinder, through which — also during the measurements — passed a constant flow of water of 15° C, taken from the thermostate. The vessels, placed on the cork-ring on the bottom of the cylinder (see Part I, fig. 2) found themselves surrounded with this thermostate water to the neck.

The results have been given in Table III.

TABLE III.

Determination of $T$ and $\Lambda$ ratios at 15° C.

| Type of oscillation | $10 \times \frac{T}{2}$ (sec) | $b_1/b_3$ | $\Lambda$ | $10 \times \frac{T}{2}$ corr.(sec) |
|---|---|---|---|---|
| Rotational | $8.10 \pm 0.024$ | $1.400 \pm 0.016$ | $0.337 \pm 0.011$ | $8.09 \pm 0.024$ |
| Meridional | $6.31 \pm 0.014$ | $1.302 \pm 0.008$ | $0.264 \pm 0.006$ | $6.30 \pm 0.014$ |
| Quadrantal | $5.12 \pm 0.021$ | $1.233 \pm 0.0043$ | $0.209 \pm 0.003$ | $5.12 \pm 0.021$ |

$$\frac{T_0}{T_1} = 1.284 \qquad \frac{\Lambda_0}{\Lambda_1} = 1.277$$

$$\frac{T_0}{T_2} = 1.580 \qquad \frac{\Lambda_0}{\Lambda_2} = 1.612$$

It will be seen that in this case, where heat exchange with the surrounding air had been prevented, there is a satisfactory agreement with the theoretical values, in particular for $T_0/T_1$ and $\Lambda_0/\Lambda_1$. The other two quotients, $T_0/T_2$, $\Lambda_0/\Lambda_2$ still are slightly too high.

In this connection we think it useful to mention some measurements of the ratio $T_0/T_2$, performed under conditions which prevented heat exchange with the surroundings, in an earlier part of this research [9]). They refer to an oleate system of slightly different composition; the oscillations were excited with the aid of the pendulum apparatus (see Part I, fig. 3). The vessels were wholly immersed in a thermostate of 15° C. The results of four series of measurements have been given in Table IV.

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[9]) These measurements were made before we knew anything about the theoretical formulae.

TABLE IV.

Determinations of $T_0/T_2$ at 15°.

| Nominal volume of the spherical vessel in c.c. | $10 \times \frac{T_0}{2}$ (sec.) | $10 \times \frac{T_2}{2}$ (sec.) | $T_0/T_2$ |
|---|---|---|---|
| 1500 | 8.64 | 5.49 | 1.574 |
| 750 | 7.66 | 4.87 | 1.573 |
| 750 | $7.63 \pm 0.04$ | $4.90 \pm 0.02$ | **1.557** |
| 500 | $6.48 \pm 0.02$ | $4.18 \pm 0.03$ | **1.550** |

The results for the first two cases (nominal volumes 1500 and 750 $cm^3$) are deduced each from ten determinations of $10 \times T/2$; those for the last two cases (nominal volumes 750 and 500 $cm^3$) have been deduced from 50 and 60 determinations of $10 \times T/2$ respectively and the mean errors are very small. The values of the ratio $T_0/T_2$ obtained in these cases (1.550, 1.557) are even nearer to the theoretical value 1.557 than the value 1.580 given in Table III [10]).

Hence judging by the most reliable measurements we come to the conclusion that the experimental values of the ratios $T_0/T_1$, $T_0/T_2$, $A_0/A_1$, $A_0/A_2$ are very near to the theoretical ones.

This proves that the theoretical formulae, deduced by BURGERS, are valid for the oleate system to which the measurements refer (1.2 % oleate, with 1.52 N KCl and 0.08 N KOH), and that the elastic behaviour of this system can be described by means of two constants; the shear modulus $G$ and the relaxation time $\lambda$.

### 6. *Shear modulus and relaxation time as functions of the temperature.*

The values of $G$ and $\lambda$, calculated with the aid of BURGERS' formulae, assuming the value $\varrho = 1.074$ for the density, have been inserted into the last columns of Tables I and II. The overall mean values of these quantities are found to be:

at 15° C: $G = 45.2$ dynes/$cm^2$; $\lambda = 2.40$ sec.
at 23° C: $G = 46.7$ dynes/$cm^2$; $\lambda = 1.08$ sec.

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[10]) The value 1.580 has been obtained by using the turning table. It may be that this method of exciting the oscillations is the cause of the abnormal values of $T_0/T_2$ and in particular of $A_0/A_2$ in Table III. The exact measurement of quantities referring to the quadrantal oscillation is difficult, when it is markedly combined with the rotational oscillation, as the air bubbles describe complicated curves instead of straight lines. It is in particular the determination of $b_1/b_3$ which is influenced in this way. The application of the pendulum apparatus for exciting the oscillations has the advantage that with a sufficiently strong impulse the rotational oscillation appears to be nearly suppressed, so that the quadrantal oscillation is much better obversable (see Part I, section 6).

Increase of temperature appears to have a small effect on the shear modulus and a much larger effect on the relaxation time.

In view of the remarkable form of the $b_1/b_3$-curve, described in Part I, it is of interest to re-calculate the values for the series of experiments considered there, so as to obtain the corresponding values of $G$ and $\lambda$. As the values of $10 \times T/2$ had been determined at temperatures not equal to those to which refer the values of $b_1/b_3$, we have drawn smooth curves through the experimental points, from which we have read off the values of the period and of the damping ratio for a series of temperatures (2.5°; 5°; 7.5°, etc.). These values have been collected in Table V and the values calculated for G and $\lambda$ have been added.

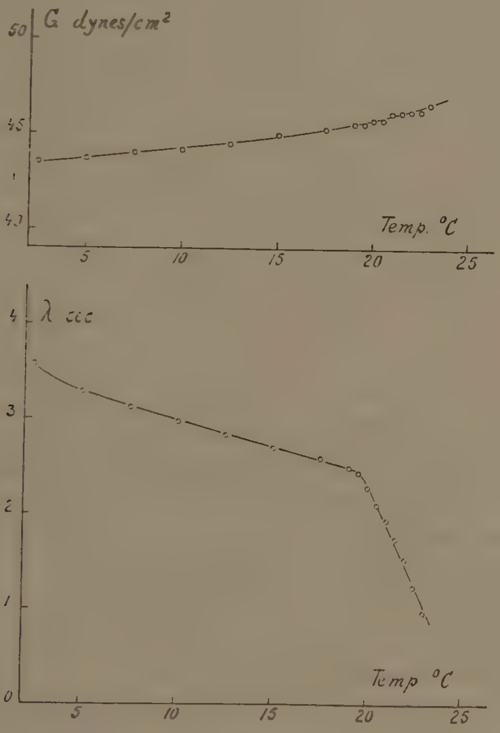

Fig. 6.

It is again found (compare fig. 6) that there is a slight increase of $G$ with temperature, whereas there is a marked decrease of $\lambda$. It is remarkable, however, that the decrease of $\lambda$ occurs in two stages: the curve exhibits two branches, joining with a relatively sharp bend. Hence there is a definite temperature (19.5° C) at which the behaviour of $\lambda$ changes in such a way that its temperature coefficient suddenly increases. There is no indication of a similar change in the curve for $G$.

It seems premature to attempt an interpretation of the course of the curves for $G$ and $\lambda$ as functions of the temperature. A much more extended

TABLE V.

Shear modulus and relaxation time as functions of the temperature.

| Temp. °C | $10 \times T/2$ (sec.) | $b_1/b_3$ | $\Lambda$ | $10 \times T/2$ corr. (sec.) | $\lambda$ (sec.) | G (dynes/cm²) |
|---|---|---|---|---|---|---|
| 2.5 | 5.57 | 1.170 | 0.157 | 5.57 | 3.57 | 43.5 |
| 5 | 5.56 | 1.185 | 0.170 | 5.56 | 3.28 | 43.7 |
| 7.5 | 5.54 | 1.195 | 0.178 | 5.54 | 3.11 | 44.0 |
| 10 | 5.53 | 1.205 | 0.186 | 5.53 | 2.97 | 44.2 |
| 12.5 | 5.51 | 1.215 | 0.194 | 5.51 | 2.83 | 44.5 |
| 15 | 5.48 | 1.225 | 0.203 | 5.48 | 2.70 | 45.0 |
| 17.5 | 5.46 | 1.235 | 0.211 | 5.46 | 2.59 | 45.3 |
| 19 | 5.44 | 1.245 | 0.219 | 5.44 | 2.49 | 45.6 |
| 19.5 | 5.44 | 1.250 | 0.223 | 5.44 | 2.44 | 45.6 |
| 20 | 5.43 | 1.270 | 0.239 | 5.43 | 2.28 | 45.8 |
| 20.5 | 5.43 | 1.295 | 0.258 | 5.43 | 2.10 | 45.8 |
| 21 | 5.42 | 1.325 | 0.281 | 5.41 | 1.93 | 46.2 |
| 21.5 | 5.42 | 1.365 | 0.311 | 5.41 | 1.74 | 46.2 |
| 22 | 5.41 | 1.425 | 0.354 | 5.40 | 1.53 | 46.3 |
| 22.5 | 5.41 | 1.550 | 0.438 | 5.40 | 1.23 | 46.3 |
| 23 | 5.40 | 1.750 | 0.559 | 5.38 | 0.96 | 46.7 |

material will be necessary, allowing the discussion of the influence of a greater number of parameters.

In the next communication we shall consider the influence of the oleate concentration.

### *Summary of Part II.*

1. Measurements of the period and the logarithmic decrement of the rotational, the meridional and the quadrantal oscillations have been executed with an 1.2 % oleate system (containing 1.52 N KCl + 0.08 N KOH), at temperatures of 15° and 23° C. The dependence of these quantities on the radius of the vessel has been investigated.

2. It has been found that both the period and the logarithmic decrement are proportional to the radius of the vessel, which proves that at both temperatures the damping of the elastic oscillations of this oleate system must be ascribed to relaxation of the elastic stresses, characterized by a single and constant relaxation time.

3. Theoretical formulae developed by BURGERS make it possible to calculate the shear modulus and the relaxation time. The shear modulus is only slightly dependent on the temperature: $G = 45$ dyne/cm² at 15° and 47 dyne/cm² at 23° C. There is a larger change in the relaxation time: $\lambda = 2.4$ sec at 15° and 1.1 sec at 23° C.

4. Using subscripts 0, 1 and 2 in order to distinguish between quantities referring to the rotational, the meridional and the quadrantal oscillations respectively, it has been found that the experimental values of the ratios $T_0/T_1$, $\Lambda_0/\Lambda_1$, $T_0/T_2$, $\Lambda_0/\Lambda_2$ are very near to the theoretical values

$T_0/T_1 = A_0/A_1 = 1.28$ and $T_0/T_2 = A_0/A_2 = 1.56$, the differences being of the order 1 or 2 %.

5. Making use of results of experiments already mentioned in Part I, curves have been constructed giving the course of $G$ and $\lambda$ with the temperature. The shear modulus presents a slight gradual increase with temperature. The relaxation time decreases; the curve exhibits two straight, branches, joining at ca. 19° C with a not very marked bend. Above 20° the temperature coefficient of $\lambda$ is about 5 or 6 times as large as below 19°.

**Zoology.** — *On the influence of lithium chloride on the eggs of Limnaea stagnalis at the 24-cell stage.* By CHR. P. RAVEN and SELMA DUDOK DE WIT. (Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of December 18, 1948.)

When the eggs of *Limnaea stagnalis* are treated with weak solutions of LiCl, characteristic malformations are induced (RAVEN 1942). On the one hand, part of the eggs may exogastrulate, giving rise to pear- or dumb-bell-shaped vesicular embryos; on the other hand, embryos with various malformations of the head region are produced. It has been proved (RAVEN 1947) that these head malformations are due to a reduction of dorsomedian parts of the head, which are derived, in normal development, from cells adjacent to the original animal pole of the egg. It was concluded from these observations that the influence of lithium chloride can be explained by its action on a polar gradient field.

RAVEN, KLOEK, KUIPER and DE JONG (1947) showed that the action of LiCl on *Limnaea* eggs is phase-specific. There is a distinct maximum of sensibility for the production of exogastrulae shortly before and during second cleavage. With regard to the production of head malformations, a first period of sensibility exists immediately after laying; a second maximum is found at the 24-cell stage. RAVEN and RIJVEN (1948) demonstrated that the susceptibility for the induction of head malformations rises gradually from the 12-cell stage until 2—3 hours after the beginning of the 24-cell stage; then, a drop in susceptibility occurs. When the eggs are treated more than 8 hours after the beginning of the 24-cell stage, no head malformations are produced.

DE GROOT (1948) studied the immediate effects of the exposure of decapsulated *Limnaea* eggs at the uncleaved stage to stronger solutions of LiCl. Hypertonic LiCl solutions caused various disturbances of the nuclear and mitotic apparatus, leading to an arrest of development at different stages of maturation and fertilization. In isotonic LiCl solutions, the nuclear processes proceeded in a normal way, but the distribution of the subcortical protoplasm was very abnormal. The formation of the animal pole plasm was suppressed at all concentrations studied.

In view of these results it appeared interesting to study the immediate effects of a treatment of decapsulated eggs at the 24-cell stage with LiCl solutions of various concentrations, in order to compare the behaviour of the eggs at both stages at which head malformations can be easily induced by LiCl treatment.

## *Material and methods*

Egg-masses were followed in their development till the 24-cell stage was reached. 2—3 Hours after the beginning of this stage, each egg-mass was divided into 6 equal parts. The eggs contained in 5 of these parts were decapsulated and transferred to solutions of 0.4 %, 0.2 %, 0.1 %, 0.05 % LiCl and to distilled water, respectively. After an exposure of 1, 2 or 3 hours, the eggs were fixed. Those of the 6th part of the egg-mass, which had remained in the capsules meanwhile, now were decapsulated too and fixed immediately; they served as controls. Hence, the eggs were fixed 3¼—5¾ hours after the beginning of the 24-cell stage.

All eggs were fixed in BOUIN's fluid and cut in sections of 5 $\mu$ thickness, which were stained either with iron haematoxylin and eosin or with azan.

## *Results*

Already at a superficial inspection of the sections it becomes clear that the treatment of the eggs with LiCl solutions or distilled water has produced peculiar changes in the size and position of the nuclei.

In normal eggs of this stage (fig. 1), the nuclei are situated at the boundary between ecto- and endoplasm in all cells. As the layer of ectoplasm at the animal side of the egg is somewhat thicker the nuclei of the animal cells are lying at a greater distance from the surface than those of the vegetative cells. Moreover, there is a clear difference in size, the animal nuclei being, on an average, distinctly smaller than the vegetative ones (4 small cells surrounding the vegetative pole, probably the 3d micromeres, have, however, still smaller nuclei).

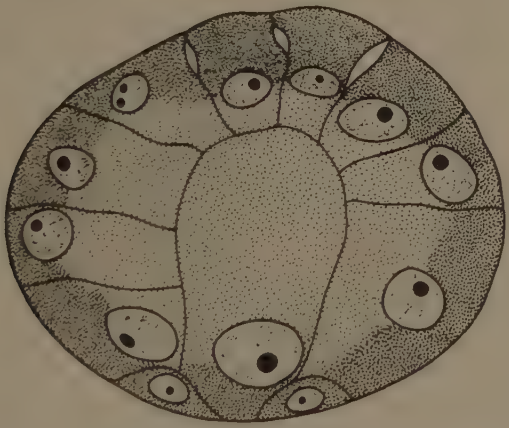

Fig. 1. *Limnaea stagnalis*. 24-cell stage. Normal egg.

In the eggs treated with distilled water and with hypotonic solutions of LiCl (0.05 and 0.1 %), the size of the nuclei has considerably increased. Moreover, they have been displaced towards the surface. In many cells, both animal and vegetative, the nuclei are lying in immediate contact with the cell surface (fig. 2); sometimes, especially in vegetative cells, the surface is (in the sections) even slightly bulging at this place. This change of position is not due to a disappearance or reduction of the layer of ectoplasm, but the nuclei have left their position at the boundary of the plasma layers and are now situated wholly or for the greater part in the ectoplasm.

In isotonic (0.2 %) or hypertonic (0.4 %) solutions of LiCl, the nuclei are smaller, but their position is the same as in hypertonic solutions.

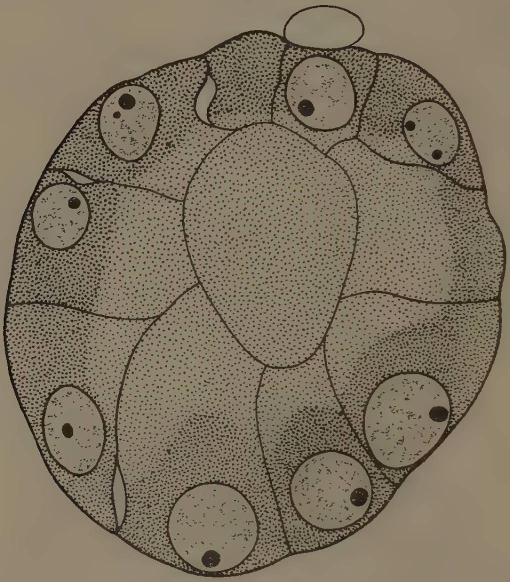

Fig. 2. *Limnaea stagnalis.* 24-cell stage. 0.05 % LiCl.

In order to get accurate data both on the swelling and the displacement of the nuclei in the different solutions, measurements have been made by means of an ocular micrometer, with oil immersion. The diameter of each nucleus was determined in 2 perpendicular directions; the product of these 2 measurements is indicated below as "nuclear value" (in $\mu^2$). Moreover, the distance between the outer side of the nuclear membrane and the cell surface (in $\mu$) is called "nuclear distance". The nuclei of the animal and vegetative half of the egg have been treated separately.

4 Egg-masses have been used for these measurements; they are indicated as series 1, 2, 3 and 4. In total, 832 nuclei have been measured.

*Nuclear values.*

The inspection of the nuclear values obtained in the controls shows that series 1 and 2 agree within the limits of error in the size of their nuclei; the same holds true as regards series 3 and 4, which show comparable nuclear values, but lower than those of series 1 and 2. Therefore, in the following table series 1 and 2, on the one hand, and 3 and 4, on the other, are taken together.

Table I shows the nuclear values (in $\mu^2$) of controls and treated eggs.

TABLE I.

| Treatment | Series (1 + 2) | | | | Series (3 + 4) | | | |
|---|---|---|---|---|---|---|---|---|
| | Animal cells | | Vegetative cells | | Animal cells | | Vegetative cells | |
| | *n* | *M* ± *m* | *n* | *M* ± *m* | *n* | *M* ± *m* | *n* | *M* ± *m* |
| Controls | 54 | 120.5 ± 5.89 | 58 | 169.0 ± 8.98 | 72 | 92.9 ± 4.71 | 46 | 122.1 ± 8.67 |
| Dist. water | 15 | 119.4 ± 10.05 | 18 | 193.3 ± 15.37 | 17 | 132.3 ± 8.29 | 26 | 226.8 ± 10.43 |
| 0.05 % LiCl | 17 | 190.3 ± 18.72 | 34 | 246.6 ± 13.03 | 58 | 172.1 ± 7.23 | 68 | 216.8 ± 5.82 |
| 0.1 % ,, | 39 | 196.2 ± 9.21 | 28 | 253.6 ± 14.50 | 35 | 136.8 ± 8.23 | 37 | 177.8 ± 8.59 |
| 0.2 % ,, | 37 | 123.6 ± 7.80 | 24 | 171.3 ± 9.46 | 9 | 128.8 ± 10.89 | 23 | 178.4 ± 11.48 |
| 0.4 % ,, | 32 | 127.4 ± 6.40 | 35 | 178.2 ± 8.80 | 16 | 118.8 ± 11.81 | 33 | 177.0 ± 8.26 |

In order to judge the significance of the differences found, for each relevant comparison a *t* has been computed; for the sake of brevity, these values are omitted here, but they have been used to draw the following conclusions (probabilities below 0.01 have been considered as significant).

In the controls a significant difference in size between animal nuclei (A) and vegetative nuclei (V) exists, the nuclear values of V being about 1.3—1.4 times those of A; hence, the volumes of the vegetative nuclei are about 1.5—1.6 times as large.

In distilled water in series (3 + 4) a considerable swelling of the nuclei has taken place, especially at the vegetative side. In series (1 + 2) this swelling is either insignificant (V) or it is lacking altogether (A).

In 0.05 % LiCl the nuclei are considerably swollen; this swelling is much stronger than in distilled water, except in (3 + 4) V, where the nuclear values show no significant difference in both cases.

In 0.1 % LiCl the general swelling of the nuclei is still obvious. In series (1 + 2) the nuclear values agree with those in 0.05 % LiCl; in series (3 + 4) they are lower, in A they have reached again the value of distilled water, in V they are even significantly lower than this.

In 0.2 % and 0.4 % LiCl the swelling has decreased further. In series (1 + 2) the nuclear values of both A and V agree with those of the controls; in series (3 + 4), they are still higher than in the controls. It is very remarkable that the animal nuclei as well as the vegetative nuclei of both series agree in size in 0.2 % (isotonic) and 0.4 % (hypertonic) solutions (nuclear values: animal cells about 125 $\mu^2$, vegetative cells about 175 $\mu^2$), notwithstanding the initial size difference between both series in the controls and the different osmotic pressures of the solutions.

Though the behaviour of both series is not altogether identical, we may summarize the results as follows: the swelling of the nuclei is greatest in 0.05 % LiCl, somewhat less in 0.1 % and distilled water, whereas the size of the nuclei is still less in 0.2 % and 0.4 % LiCl, but may be still greater than in the controls in egg capsule fluid. Considering that the egg capsule fluid is considerably hypotonic to the eggs (RAVEN and KLOMP 1946), it is clear that the size of the nuclei is not only governed by osmotic forces.

*Nuclear distances.*

Table II gives the nuclear distances (in $\mu$) of controls and treated eggs.

TABLE II.

| | Treatment | Series 1 | | Series 2 | | Series 3 | | Series 4 | |
|---|---|---|---|---|---|---|---|---|---|
| | | $n$ | $M \pm m$ | $n$ | $M \pm m$ | $n$ | $M \pm m$ | $n$ | $M \pm m$ |
| *a.* Animal nuclei | Controls | 22 | 6.50±0.475 | 32 | 5.29±0.464 | 36 | 4.55±0.690 | 36 | 6.22±0.666 |
| | Dist. water | 7 | 12.80±3.032 | 8 | 5.78±0.413 | 17 | 0.83±0.242 | — | — |
| | 0.05% LiCl | 4 | 5.95±0.670 | 13 | 0.97±0.401 | 31 | 0.32±0.125 | 27 | 1.67±0.246 |
| | 0.1% „ | 26 | 6.83±0.516 | 13 | 1.19±0.310 | 8 | 1.93±0.875 | 27 | 3.58±0.657 |
| | 0.2% „ | 6 | 14.00±3.313 | 32 | 1.79±0.182 | 9 | 0.00±0.000 | — | — |
| | 0.4% „ | 10 | 5.32±0.952 | 22 | 2.80±0.357 | 3 | 0.94±0.469 | 13 | 1.40±0.449 |
| *b.* Vegetative nuclei | Controls | 26 | 3.12±0.808 | 32 | 2.10±0.177 | 21 | 1.06±0.396 | 25 | 5.43±0.643 |
| | Dist. water | 8 | 1.92±0.834 | 10 | 1.12±0.280 | 26 | 0.38±0.147 | — | — |
| | 0.05% LiCl | 18 | 1.16±0.325 | 16 | 0.87±0.177 | 32 | 0.27±0.099 | 37 | 1.29±0.226 |
| | 0.1% „ | 16 | 1.81±0.725 | 12 | 0.70±0.273 | 7 | 0.20±0.198 | 30 | 2.06±0.505 |
| | 0.2% „ | — | — | 23 | 0.42±0.140 | 23 | 0.49±0.188 | — | — |
| | 0.4% „ | 11 | 1.40±0.462 | 24 | 1.75±0.283 | 6 | 0.70±0.314 | 27 | 2.65±0.570 |

Also in this case the significance of differences has been tested by computation of $t$ values.

In general, the nuclear distances show regular changes with the treatment; only the animal nuclei of series 1 exhibit large and quite irregular differences, which are, however, not significant, and may be largely due to the small numbers of nuclei measured in 3 of the 6 groups. Therefore, series 1 A is not further considered below.

In the controls, the nuclear distances of animal nuclei are greater than those of vegetative nuclei; except in series 4, these differences are significant.

In distilled water, in 4 out of 5 possible comparisons, the nuclear distances have diminished as compared with the controls; in 2 cases (series 3 A and 2 V), this decrease is significant.

In 0.05 % LiCl, in all 7 cases the nuclear distances have diminished as compared with the controls; in 5 cases (2—4 A, 2 V and 4 V) the decrease is significant. In all 5 possible comparisons with distilled water, the nuclear distances have further diminished in 0.05 % LiCl; this decrease is significant in 2 A.

In 0.1 % LiCl, in all 7 cases the nuclear distances are lower than those of the controls (significant difference in 2 A and V and 4 A and V), in 4 out of 5 cases lower than those in distilled water (significantly in 2 A). However, in 5 out of 7 possible comparisons they are higher than in 0.05 % LiCl; in 4 A this difference is significant.

In 0.2 % LiCl, in all 4 possible comparisons the nuclear distances have diminished with respect to the controls; the difference is significant in 3 cases (2 A and V, 3 A). In 3 out of 4 cases they are lower than in distilled water (significantly in 2 A and 3 A). No significant differences exist in comparison with 0.1 % LiCl.

In 0.4 % LiCl, in all 7 cases the values are lower than those of the controls; the difference is significant in 4 cases (2—4 A, 4 V). In 2 A, the nuclear distance is also significantly lower than in distilled water. However, in all 4 possible comparisons they are higher than those in 0.2 % LiCl; this difference is significant in series 2 V.

Summarizing, it can be said that the nuclear distances are smallest in 0.05 % LiCl (on an average: about 1 $\mu$), somewhat greater in 0.1 % and 0.2 % LiCl, still greater in 0.4 % LiCl and distilled water. Again, it may be concluded that the decrease of the nuclear distances is not directly related to the osmotic pressure of the solutions.

## *Discussion*

The treatment of decapsulated *Limnaea* eggs at the 24-cell stage with distilled water or solutions of LiCl results in 1°. an increase in the size of the nuclei, 2°. a decrease of their distance from the cell surface. Both phenomena are most pronounced in 0.05 % LiCl, and diminish in intensity with increasing concentration of LiCl. However, they cannot be explained by osmotic forces only, since in eggs treated with distilled water the swelling of the nuclei is less and their distance from the surface is greater than in 0.05 % LiCl.

It may be asked if the decrease in the distance between nuclear membrane and cell surface is due to a real displacement of the nucleus, or is only a consequence of its swelling, the centre of the nucleus remaining at the same place. From the figures of table I the average increase of the nuclear radius at various concentrations can be computed; by comparing this with the decrease of the nuclear distance, an answer to this question may be obtained.

These comparisons show that e.g. in 0.05 % LiCl the radius of the animal nuclei have increased, on an average, with $1\frac{1}{2} — 1\frac{3}{4}$ $\mu$ as compared with the controls; at the same time, the nuclear distances have decreased with $4 — 4\frac{1}{2}$ $\mu$. Hence, it appears that at least the animal nuclei undergo a real displacement towards the surface. At the vegetative side, this is less evident; the decrease of the nuclear distances with, on an average, but 1 — 2 $\mu$ may be accounted for by a similar increase in nuclear radius; only in series 4 a real displacement of the vegetative nuclei may have taken place.

This shift in the position of the nuclei towards the outer part of the cells resembles the displacement of the nuclei in neural tube, notochord and somites of *Triton* embryos which LEHMANN and ANDRES (1948) have observed after treatment of the embryos with phenol solutions.

Our results do not prove that LiCl at the 24-cell stage of *Limnaea* influences the polar (animal-vegetative) gradient-field. Rather, the displacement of the nuclei might be explained by a disturbance of an extero-interior gradient-field determining their position in normal development (cf. RAVEN 1946, p. 424). To be sure, in normal cleavage stage embryos the animal nuclei are further removed from the surface than the vegetative ones; this might be explained as being due to an interference between the extero-interior and animal-vegetative gradient-field, probably through the medium of the unequal distribution of ecto- and endoplasm over the cleavage cells. As table II shows, in Li-treated embryos this difference in "nuclear distances" between animal and vegetative cells has nearly disappeared. Moreover, as stated above, a distinct shift in the nuclei is only indicated in the animal cells, whereas in the vegetative cells the reduction of the nuclear distance may be entirely due to swelling. Hence, the disturbance of the extero-interior gradient is more pronounced at the animal side, and the Li-effect leads to a weakening of the manifestations of the animal-vegetative polarity. Whether this may be interpreted as an actual weakening of the polar gradient-field, is not certain.

## *Summary*

1. Decapsulated eggs of *Limnaea stagnalis* at the 24-cell stage have been treated with distilled water and 0.05 %, 0.1 %, 0.2 % and 0.4 % LiCl solutions.

2. This treatment leads to 1. an increase in the size of the nuclei, 2. a decrease of their distance from the cell surface. Both phenomena are most pronounced in 0.05 % LiCl, and diminish in intensity with increasing concentrations of LiCl; in distilled water they are less pronounced, too.

3. At the animal side an actual shift of the nuclei towards the surface takes place; in the vegetative cells the decrease of the distance between nucleus and cell surface may be accounted for, in most cases, by the increase in nuclear radius.

4. The displacement of the nuclei may be explained by a disturbance of an extero-interior gradient-field, which is most pronounced at the animal side.

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## **Mathematics.** — *On the symbolical method.* II. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 27, 1948.)

### § 5. *Six points on a conic.*

It is generally accepted that PASCAL discovered his theorem before the year 1644. Not before 1806 BRIANCHON found the dual theorem and afterwards lead the work of STEINER (1828), KIRKMAN (1849), CAYLEY–SALMON (1849) to the discovery of the VERONESE-properties of Hexagramma mysticum by VERONESE (1877 — Atti Linc., 3—I, 649—703 seq.). Not only the long period elapsed during this development but also the fact that prominent mathematicians as STEINER, HESSE, SCHRÖTER made statements and suggestions which were proved to be false, indicates that from a geometrical point of view these VERONESE-properties do not belong to the simplest ones. CREMONA's discovery of the relation between Hexagramma mysticum and the lines on a cubic surface with a conical point lead to a more-dimensional treatment of the problem.

In the following it will be shown, that, the general method being founded, and ternary geometry being reduced to binary geometry, the VERONESE-properties are indeed the simplest theorems one can write down, the PASCAL-theorem being proved.

I. PASCAL's theorem.

$$(lp)(km)(ni)P_{ik,mn} + (mi)(ln)(pk)P_{kl,np} + (nk)(mp)(il)P_{lm,pi} \equiv 0$$

*i, k, l, m, n, p* being six parameters of points on a conic.

Proof.

$$(\mathbf{lp})(km)\,\mathbf{a}_i a_n + (\mathbf{lp})(in)\,a_k a_m + (mi)(ln)\,a_k a_p + (\mathbf{mi})(\mathbf{kp})\,a_l a_n +$$
$$+ (nk)(mp)\,a_l a_l + (nk)(li)\,a_m a_p \equiv$$
$$-(il)(\mathbf{k}m)\,a_p a_n + (mp)(in)\,a_k a_l - (ml)(in)\,a_k a_p + (mi)(ln)\,a_k a_p +$$
$$+ (\mathbf{ki})(mp)\,a_l a_n + (nk)(mp)\,a_l a_l + (nk)(li)\,a_m a_p \equiv$$
$$[(mi)(ln) - (ml)(in) - (il)(mn)]\,a_p a_k \equiv 0.$$

In virtue of the relation $P_{ik,mn} \equiv -P_{mn,ik}$ this relation is invariant under cyclical permutation and reversal of the order $iklmnp$. So we have 60 PASCAL-lines $p_{iklmnp}$.

II. $p_{iklmnp} = (li)(pk)\,\alpha_n\alpha_m - (ln)(pm)\,\alpha_i\alpha_k = 0.$

Proof: Breaking up the ternary brackets into binary cycles we have from

$$(ikl)(knp)(mnx) + (nkl)(mnp)(ikx) = 0,$$

dividing by $(ik)(kl)(kn)(np)(mn)$ the equation stated in the theorem From this it follows immediately:

$$p_{iklmnp} \equiv p_{pnmlki} \equiv -p_{klmnpi}.$$

Transforming $a_m$ in $(li)$ and $a_k$ in $(ln)$ we obtain

$$p_{iklmnp} \equiv (mi)(pk)\,a_n a_l + (pl)(km)\,a_n a_i + (mp)(kn)\,a_i a_l,$$

giving the coordinates on the fundamental triangle $i, l, n$ etc.

III. The three PASCAL-lines

$$p_{imknlp}, \quad p_{inkplm}, \quad p_{ipkmln},$$

are concurrent in the STEINER-point

$$S_{ikl,\,mnp} \equiv \Delta_{ikl} - \Delta_{mnp} = 0.$$

Proof: $S_{ikl,\,mnp} \equiv \{ik\} + \{kl\} + \{li\} + \{nm\} + \{pn\} + \{mp\} \equiv$

$$\begin{aligned} &\equiv P_{in,\,mk} + P_{kp,\,nl} + P_{lm,\,pi} \equiv \\ &\equiv P_{ip,\,nk} + P_{km,\,pl} + P_{ln,\,mi} \equiv \\ &\equiv P_{im,\,pk} + P_{kn,\,ml} + P_{lp,\,ni}. \end{aligned}$$

The first three PASCAL-points are on $p_{ipkmln}$, the second three on $p_{imknlp}$ and the third three on $p_{inkplm}$.

As $S_{ikl,\,mnp} \equiv -S_{mnp,\,ikl}$ and because of the cyclical symmetry in $ikl$ and $mnp$, there are 20 STEINER-points.

A triangle formed by $[ik] = 0$, $[lm] = 0$, $[np] = 0$ is called a VERONESE-triangle $(ik, lm, np)$ if all $i, k, l, m, n, p$ are different. There are 15 VERONESE-triangles.

IV. The 15 VERONESE-triangles are in 20 triples perspective with a STEINER-point as centre.

Proof: We can split up $S_{ikl,\,mnp}$ in three other ways:

$$\begin{aligned} S_{ikl,\,mnp} &\equiv P_{in,\,mk} + P_{km,\,pl} + P_{lp,\,ni} \equiv \ldots\ldots \quad (1) \\ &\equiv P_{lm,\,pi} + P_{ip,\,nk} + P_{nk,\,lm} \equiv \ldots\ldots \quad (2) \\ &\equiv P_{kp,\,nl} + P_{ln,\,mi} + P_{im,\,pk} \equiv \ldots\ldots \quad (3) \end{aligned}$$

which gives us the vertices of the VERONESE-triangles

$(in, mk, pl)$, $(lm, ip, nk)$ and $(kp, ln, mi)$.

Intersecting corresponding sides of:

(1) *and* (2): $P_{mk,\,ip}$, $P_{lp,\,kn}$, $P_{ni,\,ml}$ on $p_{iplmkn}$,
(1) *and* (3): $P_{km,\,nl}$, $P_{lp,\,im}$, $P_{in,\,kp}$ on $p_{inlpkm}$,
(2) *and* (3): $P_{ip,\,nl}$, $P_{nk,\,im}$, $P_{ml,\,pk}$ on $p_{imlnkp}$,

these three PASCAL-lines are the axis of perspectivity of the pairs of triangles and concurrent in $S_{ikl,\,mpn}$, the conjugate STEINER-point of $S_{ikl,\,mnp}$.

V. Two conjugate STEINER-points form the JACOBIAN (common harmonical pair) of $\Delta_{ikl}$, $\Delta_{mnp}$ and the points of intersection of the line joining them with the conic.

Proof: $$S \equiv S_{ikl,mnp} \equiv \Delta_{ikl} - \Delta_{mnp} = 0$$ $$S_1 \equiv S_{ikl,mpn} \equiv \Delta_{ikl} + \Delta_{mnp} = 0$$
from which the first part is evident.

$(\Omega' S)(\Omega' S_1) \equiv (\Omega' \Delta_{ikl})^2 - (\Omega' \Delta_{mnp})^2 = 0$, as $(\Omega' \Delta_{ikl})^2$ is independent of the indices $i, k, l$, from which the second half follows.

VI. The linear relation between the three in $S_{iln,kmp}$ concurrent PASCAL-lines is

$$p_{iklmnp} + p_{imlpnk} + p_{iplknm} \equiv 0.$$

Proof: $$p_{iklmnp} \equiv (li)(pk) a_n a_m - (ln)(pm) a_i a_k$$ $$p_{imlpnk} \equiv (li)(km) a_n a_p - (ln)(kp) a_i a_m$$ $$p_{iplknm} \equiv (li)(mp) a_n a_k - (ln)(mk) a_i a_p$$

Adding these three forms we obtain according to the fundamental identity $(pk) a_m \equiv (mk) a_p - (mp) a_k$ etc. a sum $\equiv 0$.

VII. $$S_{ikl,mnp} \equiv S_{ikp,mnl} + S_{inl,mkp} + S_{mkl,inp}.$$

Proof: Inserting the brackets $\{ik\}$ ... etc. all terms cancel.

VIII. $$\langle iln \rangle \langle kmp \rangle S_{iln,kmp} + \langle kln \rangle \langle imp \rangle S_{kln,imp} + \\ + \langle imn \rangle \langle klp \rangle S_{imn,klp} + \langle ilp \rangle \langle kmn \rangle S_{ilp,kmn} \equiv 0.$$

Proof: Consider the points

$$S_{imn,klp} \quad \text{and} \quad \lambda S_{iln,kmp} + \mu S_{kln,imp}.$$

Calculating the linear-factors for the line $p_{iklmnp}$ and the second point the coefficient of $\lambda$ vanishes, that of $\mu$ is, as the 12 terms cancel nearly all

$$2 \frac{\langle imn \rangle \langle kpl \rangle - \langle kmn \rangle \langle ipl \rangle}{(kl)(ip)(mn)},$$

whereas the linear-factor of the first point is

$$2 \frac{\langle ipl \rangle \langle nkm \rangle - \langle lnk \rangle \langle pmi \rangle}{(kl)(ip)(mn)}.$$

Again, with $p_{kilmnp}$ we find the coefficient of $\mu$ vanishing, that of $\lambda$ being

$$2 \frac{\langle imn \rangle \langle kpl \rangle - \langle kmn \rangle \langle ipl \rangle}{(il)(kp)(mn)},$$

whereas the first point gives

$$2 \frac{\langle ilp \rangle \langle knm \rangle - \langle iln \rangle \langle mkp \rangle}{(il)(kp)(mn)}.$$

The three points $S_{imn,klp}$, $S_{iln,kmp}$, $S_{kln,imp}$ are therefore collinear the relation being

$$[\langle ilp\rangle\,\langle mkn\rangle - \langle iln\rangle\,\langle mkp\rangle]\,S_{iln,kmp} +$$
$$[-\langle ilp\rangle\,\langle kmn\rangle - \langle lnk\rangle\,\langle pmi\rangle]\,S_{kln,imp} \equiv$$
$$[-\langle imn\rangle\,\langle klp\rangle - \langle kmn\rangle\,\langle ipl\rangle]\,S_{imn,klp}$$

from which according to theorem VII we have VIII.

*q.e.d.*

Remark: The four STEINER-points are collinear on a line of STEINER–PLÜCKER $j(ik, lm, np)$.

The process of forming the linear-factors from $p$ and $S$ will be given more in details in deducting the equations of the STEINER–PLÜCKER-lines.

IX. $j(ik, lm, np) \equiv (kl)\,(mn)\,(pi)\;p_{klmnpi} + (im)\,(lp)\,(nk)\,p_{imlpnk} = 0.$

Proof:

$j(ik, lm, np) \equiv \lambda\, p_{iklmnp} + \mu\, p_{imlpnk}$, where $\lambda : \mu$ can be calculated from the condition that this line, through $S_{iln,kmp}$ contains the point $S_{kln,imp}$ also.

Now

$$S_{kln,imp} \equiv \{kl\} + \{ln\} + \{nk\} + \{mi\} + \{pm\} + \{ip\}$$
$$p_{iklmnp} \equiv (li)\,(pk)\,a_n\,a_m - (ln)\,(pm)\,a_i\,a_k\,.$$

The linear-factor of this point and this line is given by the sum of twelve linear-factors. However eight cancel and we are left with

$$\frac{(li)\,(pk)}{(kl)}\,[(nk)\,(ml) + (nl)\,(mk)] + \frac{(li)\,(pk)}{(ip)}\,[(ni)\,(mp) + (np)\,(mi)] +$$
$$-\,(pm)\,[(il)\,(kn) + (in)\,(kl)] - (ln)\,[(ip)\,(km) + (im)\,(kp)] \equiv$$
$$2\,\frac{(li)\,(pk)\,(nk)\,(ml)}{(kl)} + 2\,\frac{(li)\,(pk)\,(np)\,(mi)}{(ip)} - 2\,(pm)\,(il)\,(kn) - 2\,(ln)\,(im)\,(kp) \equiv$$
$$2\,\frac{(li)\,(nk)\,(mk)\,(pl)}{(kl)} + 2\,\frac{(im)\,(kp)\,(ni)\,(lp)}{(ip)} \equiv$$
$$\frac{2}{(ip)\,(kl)\,(mn)}\,[\langle imn\rangle\,\langle kpl\rangle - \langle kmn\rangle\,\langle ipl\rangle]\,,$$

which gives the coefficient of $\lambda$ in the equation for $\lambda : \mu$ obtained by substituting $S_{kln,imp}$ in $j(ik, lm, np) = 0$.

The coefficient of $\mu$ becomes:

$$\frac{2}{(nk)\,(mi)\,(lp)}\,[\langle klp\rangle\,\langle inm\rangle - \langle ipl\rangle\,\langle kmn\rangle]\,,$$

which proves the equation for the STEINER–PLÜCKER-line given above.

From this we have

$-(ik)\,(lm)\,(np)\;j(ik, lm, np) \equiv\; < iklmnp >\, p_{iklmnp} + < imlpnk >\, p_{imlpnk}$
which changes sign under $k \rightleftarrows i$ and is invariant under the interchange of

*ik, lm; ik, np; lm, np,* thus showing that there are only 15 STEINER–PLÜCKER-lines forming a $(20_3, 15_4)$ configuration of three perpective triangles with the STEINER-points.

X. The three PASCAL-lines $p_{kmplin}$, $p_{mpknli}$, $p_{pkminl}$ are concurrent in the KIRKMAN-point

$$K_{iklmnp} \equiv \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\} + \{pi\}.$$

Proof:

$$\begin{aligned} K_{iklmnp} &\equiv P_{im,nk} + P_{kp,il} + P_{ln,pm} \equiv \\ &\equiv P_{il,mk} + P_{kn,pl} + P_{mp,in} \equiv \\ &\equiv P_{in,pk} + P_{km,nl} + P_{lp,im}. \end{aligned}$$

The first three points lie on $p_{impknl}$, the second three on $p_{ilpmkn}$, the third three on $p_{inlpkm}$, which proves the theorem.

Evidently

$$K_{iklmnp} \equiv K_{klmnpi} \equiv -K_{pnmlki},$$

so there are only 60 KIRKMAN-points.

On the PASCAL-lines through $K_{iklmnp}$ are six other KIRKMAN-points forming the triangles $K_{mpilkn}$, $K_{klpinm}$, $K_{lnmipk}$ and $K_{lmpnki}$, $K_{pnikml}$, $K_{mikpnl}$. The sides of these triangles are PASCAL-lines; corresponding sides meet in three KIRKMAN-points on $p_{iklmnp}$. In this way a DESARGUES-configuration $(10_3, 10_3)$ is generated. There are in all six of these decades of KIRKMAN-points and PASCAL-lines.

XI. The linear relation between the PASCAL-lines concurrent in a KIRKMAN-point is

$$p_{kmplin} + p_{pkminl} + p_{mpknli} \equiv 0.$$

Proof:

$$\begin{aligned} p_{mpknli} &\equiv (lm)(kp)\, a_n \alpha_l - (ln)(ki)\, a_m a_p \\ -p_{lnlmkp} &\equiv (ik)(pm)\, a_n \alpha_l - (il)(pn)\, a_k a_m \\ p_{impknl} &\equiv (pi)(lm)\, a_n a_k - (pn)(lk)\, \alpha_l a_m. \end{aligned}$$

Adding we obtain, carrying out the indicated identical transformations

$$a_m (ki) \left[ (pn) a_l + (lp) a_n - (ln) a_p \right] \equiv 0.$$

*q.e.d.*

XII. $$\begin{aligned} \langle iklmnp \rangle K_{iklmnp} &\equiv \langle klmnp \rangle a_i^2 + \langle ikmnp \rangle a_l^2 + \langle iklmp \rangle a_n^2 \equiv \\ &\equiv \langle ilmnp \rangle a_k^2 + \langle iklnp \rangle a_m^2 + \langle iklmn \rangle a_p^2. \end{aligned}$$

Proof: Dividing by $< iklmnp >$ and carrying out the only possible identical transformations we have

$$\frac{(pk)\, a_i^2}{(pi)(ik)} + \frac{(km)\, a_l^2}{(kl)(lm)} + \frac{(mp)\, a_n^2}{(mn)(np)} \equiv \{pi\} + \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\}$$

*q.e.d.*

XIII. $K_{iklmnp} + K_{imlpnk} + K_{iplknm} \equiv 0.$

Proof: Inserting the symbols $\{ik\}$ ...: all terms cancel.

The corresponding KIRKMAN-points are therefore collinear on a CAYLEY–SALMON-line $c_{iln,kmp}$.

Along the same line is evident:

$$K_{kmplin} + K_{pkminl} + K_{mpknli} \equiv 2\, S_{kmp,iln}$$

$$K_{kmplin} - K_{pkminl} + K_{mpknli} \equiv 2\,(P_{ml,ip} + P_{pi,ml}) \equiv \frac{2\,(ip)\,(lm)}{(im)\,(lp)} \cdot P_{ml,ip}.$$

XIV. $\langle iklmnp \rangle\, K_{iklmnp} + \langle imlpnk \rangle\, K_{imlpnk} + \langle iplknm \rangle\, K_{iplknm} \equiv$

$$\equiv \langle inl \rangle\, \langle kmp \rangle\, S_{inl,kmp}.$$

Proof:

$$-2\,\langle inl \rangle\, \langle kmp \rangle\, S_{inl,kmp} \equiv \langle kmp \rangle \{ (li)^2\, a_n^2 + (ln)^2\, a_i^2 + (ni)^2\, a_l^2 \} +$$
$$+ \langle iln \rangle\, \{ (km)^2\, a_p^2 + (mp)^2\, a_k^2 + (pk)^2\, a_m^2 \} \equiv \{ \langle kmp \rangle\, (ln)^2 + \langle kln \rangle\, (mp)^2 +$$
$$+ \langle mln \rangle\, (pk)^2 + \langle pln \rangle\, (mk)^2 \}\, a_i^2 + \ldots a_l^2 + \ldots a_n^2.$$

The left-hand-side is

$$\{ \langle klmnp \rangle + \langle mlpnk \rangle + \langle plknm \rangle \}\, a_i^2 + \ldots a_l^2 + \ldots a_n^2.$$

Now we have

$$\langle kmp \rangle\, (ln)^2 + \langle kln \rangle\, (mp)^2 + \langle mln \rangle\, (pk)^2 + \langle pln \rangle\, (mk)^2 \equiv$$
$$- \langle kmpln \rangle - \langle kmpnl \rangle - \langle klnmp \rangle - \underline{\langle klnpm \rangle}$$
$$- \langle plnmk \rangle - \langle plnkm \rangle - \underline{\langle mlnpk \rangle} - \underline{\langle mlnkp \rangle} \equiv$$
$$-2\,[\langle kmpln \rangle + \langle mpkln \rangle + \langle mkpln \rangle],$$

as follows inverting the order in the underlined cycles.

On the other hand is

$$\langle klmnp \rangle + \langle mlpnk \rangle + \langle plknm \rangle \equiv$$
$$-(lm)^2\, \langle knp \rangle - (lp)^2\, \langle mnk \rangle - (lk)^2\, \langle pnm \rangle - \langle kmlnp \rangle - \langle mplnk \rangle +$$
$$- \langle pklnm \rangle \equiv -[\langle kmpln \rangle + \langle mpkln \rangle + \langle mkpln \rangle]$$

which proves the identity.

The line of CAYLEY–SALMON $c_{iln,kmp}$ contains the point $S_{inl,kmp}$.

Starting with the PASCAL-lines through a KIRKMAN-point or through the STEINER-point on a CAYLEY–SALMON-line we obtain different, equivalent equations.

XV. $c_{iln,kmp} \equiv \langle kmplin \rangle\, p_{kmplin} + \langle mpknli \rangle\, p_{mpknli} +$

$$+ \langle pkminl \rangle\, p_{pkminl}.$$

Proof: The equation is of the form

$$\lambda\, p_{kmplin} + \mu\, p_{pkminl} = 0,$$

where $\lambda : \mu$ can be obtained by substituting the coordinates of $K_{imlpnk}$.

Now with

$$p_{pkminl} \equiv (mp)\,(lk)\,a_n\,a_l - (mn)\,(li)\,a_p\,a_k$$

$$\langle imlpnk \rangle\, K_{imlpnk} \equiv \langle mlpnk \rangle\, a_i^2 + \langle impnk \rangle\, a_l^2 + \langle imlpk \rangle\, a_n^2$$

we have:

$$(p'_{pkminl}\, K_{imnplk}) \equiv$$

$$\frac{2\,(li)\,(ki)\,(pn)\,(nk)}{\langle imlpnk \rangle}\,[-\mathbf{(mp)}\,(lk)\,(im)\,(mp)\,\mathbf{(nl)} - (mn)\,(ml)\,(lp)\,\mathbf{(km)}\,\mathbf{(pi)} +$$

$$-(mn)\,(im)\,(mp)\,(pl)\,(kl) + (mn)\,(im)\,(ml)\,(lp)\,(pk)\,] \equiv$$

$$\frac{2\,(li)\,(ki)\,(pn)\,(nk)\,(ml)\,(mp)}{\langle imlpnk \rangle}\,[-(np)\,(lk)\,(im) + (mn)\,(lp)\,(ki)\,] \equiv$$

$$2\,\frac{\langle liknmp \rangle - \langle limpnk \rangle}{(im)\,(lp)\,(nk)} = 2\,\frac{\langle limpkn \rangle - \langle linkmp \rangle}{(im)\,(lp)\,(nk)}\,.$$

Again

$$(p'_{kmplin}\, K_{imnplk}) \equiv 2\,\frac{\langle minlpk \rangle - \langle milnkp \rangle}{(im)\,(lp)\,(nk)}\,.$$

So the CAYLEY–SALMON-line is

$$-[\langle limpkn \rangle - \langle linkmp \rangle]\, p_{kmplin} + [\langle minlpk \rangle - \langle milnkp \rangle]\, p_{pkminl} \equiv$$
$$\equiv \langle kmplin \rangle\, p_{kmplin} + \langle pkminl \rangle\, p_{pkminl} - \langle mpknli \rangle\, [\, p_{kmplin} + p_{pkminl}\,]$$

from which the theorem follows.

Evidently $c_{iln,kmp} \equiv c_{lni,mpk} \equiv + c_{kmp,lni}$.

Moreover; as is clear from the geometrical point of view

**XVI.** $c_{iln,kmp} - c_{iln,mpk} \equiv 0$.

Proof: Inserting the $p_{kmplin}, \ldots$ and summing up every two terms containing the same $\alpha$-product the left-hand-side is

$$\langle kmp \rangle\, \langle lin \rangle\, [\,(nm)\,(pk)\,\alpha_i\,\alpha_l + (mp)\,(lk)\,\alpha_n\,\alpha_i + (ip)\,(km)\,\alpha_n\,\alpha_l\,] +$$

$$-\langle kmp \rangle\, \langle lin \rangle\, [\,(pi)\,(nl)\,\alpha_k\,\alpha_m + (kl)\,(in)\,\alpha_m\,\alpha_p + (mn)\,(li)\,\alpha_p\,\alpha_k\,]$$

Now: $(ip)\,\mathbf{(km)}\,\alpha_n\,\alpha_l \equiv \mathbf{(ip)}\,(nm)\,\alpha_k\,\alpha_l - (ip)\,\mathbf{(nk)}\,\alpha_m\,\alpha_l \equiv$

$$\mathbf{(lp)}\,(nm)\,a_k\,\alpha_i - (li)\,(nm)\,\alpha_k\,\alpha_p - \mathbf{(ip)}\,(lk)\,\alpha_m\,\alpha_n + (ip)\,(ln)\,\alpha_m\,\alpha_k \equiv$$

$$-(pk)\,(nm)\,\alpha_l\,\alpha_i - (kl)\,\mathbf{(nm)}\,\alpha_p\,\alpha_i - (np)\,(lk)\,\alpha_m\,\alpha_i + (in)\,(kl)\,\alpha_m\,\alpha_p +$$

$$+(li)\,(mn)\,\alpha_k\,\alpha_p + (pi)\,(nl)\,\alpha_m\,\alpha_k \equiv -(pk)\,(nm)\,\alpha_l\,\alpha_i - (kl)\,(pm)\,\alpha_n\,\alpha_l +$$

$$+(in)\,(kl)\,\alpha_m\,\alpha_p + (li)\,(mn)\,\alpha_k\,\alpha_p + (pi)\,(nl)\,\alpha_m\,\alpha_k$$

so all terms in $c_{iln,kmp} - c_{iln,mpk}$ cancel.

There are 20 Cayley-Salmon-lines.

**XVII.** $c_{iln,kmp} + c_{kln,imp} + c_{imn,klp} + c_{ilp,kmn} \equiv 0$.

Proof: Inserting the forms all terms cancel because of the identities

$$c_{iln,kmp} \equiv c_{lin,kmp}\quad , \quad p_{iklmnp} \equiv p_{pnmlki} \equiv - p_{klmnpi}.$$

XVIII. $c_{iln,kmp} \equiv \langle iknmlp \rangle \, p_{iknmlp} + \langle imnplk \rangle \, p_{imnplk} +$
$+ \langle ipnklm \rangle \, p_{ipnklm}.$

Proof: Substituting $K_{iklmnp}$ in $\lambda p_{iknmlp} + \mu p_{imnplk} = 0$ we find for the coefficient of $\lambda$ at first

$$\frac{2\,(ni)\,(pk)\,(mi)\,(lk)}{(ik)} + \frac{2\,(ni)\,(pk)\,(mi)\,(lp)}{(pi)} - \frac{2\,(nl)\,(pm)\,(kl)\,(im)}{(lm)} + \\ - \frac{2\,(nl)\,(pm)\,(kn)\,(im)}{(mn)},$$

as all other terms cancel. Multiplying by $< iklmnp >$ this form contains a factor $-2[(ik)(lm)(np) + (ip)(lk)(mn)]$ the remaining factor being

$$< imnplk > - < imlknp >.$$

Again, the coefficient of $\mu$ is found to be

$$2\,(ni)\,(km) \left[ \frac{(lk)\,(pi)}{(ik)} + \frac{(lm)\,(pn)}{(mn)} \right] - 2\,(nl)\,(kp) \left[ \frac{(ik)\,(ml)}{(kl)} + \frac{(ip)\,(mn)}{(np)} \right],$$

which multiplied by $< iklmnp >$ contains the same factor as the coefficient of $\lambda$ the remaining factor being

$$< imlnp > - < iknmlp >.$$

Therefore the form $c_{iln,kmp}$ is, apart from a constant factor $\varrho$ equal to the right-hand-side. Specialising $l \equiv m$, $p \equiv k$ we find

$$\varrho < ilnklk > p_{ilnklk} \equiv < klklin > p_{klklin}$$

which gives immediately $\varrho = 1$. q.e.d.

As in the case of the STEINER-points a second linear relation between the $c_{iln,kmp}$ could be derived in an analoguous way. Leaving apart for the moment the question wether the equation $\Sigma$ is irreducible we have directly:

XIX. The CAYLEY–SALMON-lines $c_{iln,mpk}$, $c_{ilk,nmp}$, $c_{ipn,mlk}$, $c_{mpn,ilk}$ are concurrent in the point

$$\Sigma(im,\, nk,\, lp) \equiv S_{inl,kmp} + K_{iklmnp}$$

Proof: Splitting up in { }-brackets we have

$$S_{inl,kmp} + K_{iklmnp} \equiv S_{ikl,nmp} + K_{inlmkp} \equiv S_{inp,mlk} + K_{ikpmnl} \equiv \\ \equiv S_{mnl,ipk} + K_{inpmkl} \equiv \{in\} + \{nl\} + \{li\} + \{mk\} + \{pm\} + \{kp\} + \\ + \{ik\} + \{kl\} + \{lm\} + \{mn\} + \{np\} + \{pi\}$$

q.e.d.

$\Sigma(im,\, nk,\, lp)$ is written in four ways as a sum of two points on each of the CAYLEY–SALMON-lines

Evidently the $\Sigma$, $c$ form a $(15_4, 20_3)$-configuration of two perspective tetragons c.q. three perspective tri-sides.

Writing

$$\Sigma(im, nk, lp) \equiv P_{im,kn} + P_{nk,pl} + P_{lp,mi} + P_{im,nk} + P_{kn,pl} + P_{lp,im}$$

it is evident, that $\Sigma$ is invariant under the interchange of two elements in a pair, changes sign if two pairs are commuted.

XX. $\Sigma(im, lk, np) + \Sigma(ik, lp, nm) + \Sigma(ip, lm, nk) \equiv 3\, S_{iln,kmp}$

Proof: Splitting up in { }-brackets all terms cancel.

XXI. $K_{iklmnp} \equiv P_{im,nk} + P_{kn,pl} + P_{lp,im}$

Proof: Evident; see XIX.

To each KIRKMAN-point corresponds *one* VERONESE-triangle. Every VERONESE-triangle corresponds to four KIRKMAN-points viz. $(im, nk, lp)$ corresponds to $K_{iklmnp}$, $K_{mklinp}$, $K_{inlmkp}$, $K_{ikpmnl}$, obtained from one of these by interchanging a pair of opposite indices. If we substitute the coordinates of $K_{iklmnp}$ in $\lambda a_i a_n + \mu a_k a_m = 0$ we obtain as nearly all terms cancel:

$$\frac{(il)\,(ln)\,(km)}{(kl)\,(lm)}\,\lambda - \frac{(in)\,(mp)\,(pk)}{(np)\,(pi)}\,\mu = 0,$$

from which follows, because of the symmetry in $l, p$:

XXII. $P_{in,km}$, $K_{iklmnp}$, $K_{ikpmnl}$ are collinear on the VERONESE-line

$$v\,(ik, mn) \equiv (in)\,(lk)\,(lm)\,(pk)\,(pm)\,a_i\,a_n + (km)\,(li)\,(ln)\,(pi)\,(pn)\,a_k\,a_m = 0.$$

Evidently: $v(ik, mn) \equiv v(ki, nm) \equiv -v(mn, ik) \equiv -v(nm, ki)$, so there are 90 VERONESE-lines, the sides of the 15 tetragons of KIRKMAN-points, corresponding to the 15 VERONESE-triangles.

XXIII. The PASCAL-point $P_{in,km}$ is incident with two VERONESE-lines, which are harmonical to the sides of the hexagon through $P_{in,km}$.

Proof: An interchange of $k, m$ òr $i, n$ changes $+$ in $-$ in the equation $v(ik, mn) = 0$. q.e.d.

On the six sides of the $K$-tetragon are the PASCAL-points:

$$P_{in,km},\ P_{il,mp},\ P_{ln,kp},\ P_{ik,mn},\ P_{kl,np},\ P_{lm,pi}.$$

The last three points are on $p_{iklmnp}$, whereas the other three are the vertices of a triangle with the sides $p_{ikpmnl}$, $p_{inlmkp}$, $p_{mklinp}$. So the $K$-tetragon and the corresponding $p$-tetragram form a DESARGUES $(10_3, 10_3)$ configuration.

In all fifteen of these configurations can be formed from the sixty PASCAL-lines and KIRKMAN-points, and the 90 VERONESE-lines.

XXIV. $v(ik, mn) + v(np, lk) + v(lm, ip) \equiv 0.$

Proof: Evident.

These three lines are concurrent in a point, the equation of which can be found by intersecting $v(ik, mn) = 0$ and $v(np, lk) = 0$ which according to the fundamental formulae, dividing by $-<kmp><iln>$ is

$$-\langle imnplk\rangle\, P_{im,pl} + \langle imlknp\rangle\, P_{im,kn} + \langle iknmlp\rangle\, P_{nk,pl} = 0.$$

As the left-hand-side is invariant under cyclical permutation and reversal of the order $iklmnp$ there are 60 of these points. We denote the left-hand-side by $<iklmnp>\, Z_{iklmnp}$ and we obtain dividing by $<iklmnp>$

$$-\frac{(im)\,(pl)}{(lm)\,(pi)}\,[\{il\}+\{mp\}]+\frac{(im)\,(kn)}{(ik)\,(mn)}\,[\{in\}+\{mk\}]+$$
$$+\frac{(kn)\,(lp)}{(kl)\,(np)}\,[\{nl\}+\{kp\}]\equiv$$

$\{li\}+\{pm\}+\{ml\}+\{ip\}+\{in\}+\{mk\}+\{nm\}+\{ki\}+\{nl\}+$ $+\{kp\}+\{lk\}+\{pn\}\equiv S_{inl,\,kmp}-K_{iklmnp}$. So we have:

XXV. $Z_{iklmnp}\equiv S_{inl,\,kmp}-K_{iklmnp}$.

XXVI. The six VERONESE-lines through the vertices of a VERONESE-triangle are by three concurrent in four points $Z$.

Proof: The equations of the lines through the vertices of $(ik,\,lm,\,np)$ are

$$(nl)\,(nm)\,(pl)\,(pm)\,(ik)\,a_i\,a_k+\varepsilon_1\,(ni)\,(nk)\,(pi)\,(pk)\,(lm)\,a_l\,a_m=0$$
$$(in)\,(ip)\,(kn)\,(kp)\,(lm)\,a_l\,a_m+\varepsilon_2\,(il)\,(im)\,(kl)\,(km)\,(np)\,a_n\,a_p=0$$
$$\varepsilon_3\,(nl)\,(nm)\,(pl)\,(pm)\,(ik)\,a_i\,a_k+(il)\,(im)\,(kl)\,(km)\,(np)\,a_n\,a_p=0,$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ are $\pm 1$.

The determinant of the coefficients of $a_i\,a_k$, $a_l\,a_m$, $a_n\,a_p$ is

$$D\,(1+\varepsilon_1\varepsilon_2\varepsilon_3),$$

where $D$ is the discriminant $(ik)\,(il)\,\ldots\,(ip)\,(kl)\,\ldots\,(np)$.

The four points of intersection are $Z_{imnklp}$, $Z_{ilpkmn}$, $Z_{impkln}$, $Z_{ilnkmp}$ of which each can be obtained from the others by interchanging opposite indices.

XXVII. The three points $Z_{iklmnp}$, $Z_{implnk}$, $Z_{iplknm}$ are incident with the CAYLEY–SALMON-line $c_{iln,\,kmp}$.

Proof: We have to show that $v(ik,\,nm)$, $v(np,\,lk)$, $c_{iln,\,kmp}$ are concurrent.

Now $c_{iln,\,kmp}$ can be written:

$$\langle kmplin\rangle\,[(mn)\,(ik)\,a_l\,a_p-(ml)\,(ip)\,a_n\,a_k]+$$
$$\langle mpknli\rangle\,[(pi)\,(lm)\,a_n\,a_k-(pn)\,(lk)\,a_i\,a_m]+$$
$$\langle pkminl\rangle\,[(kl)\,(np)\,a_i\,a_m-(ki)\,(nm)\,a_l\,a_p]=0.$$

The determinant of the three equations in $a_i\,a_m$, $a_k\,a_n$, $a_l\,a_p$ given by $v(ik,\,nm)=0$, $v(np,\,lk)=0$, $c_{iln,\,kmp}=0$ is therefore apart from a factor $(lk)\,(pn)\,(lm)\,(pi)\,(ik)\,(mn)\cdot(im)\,(ln)\,(pk)\cdot(kn)\,(li)\,(pm)\cdot(pl)\,(im)\,(nk)$ equal to

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ [(km)\,(in)\,(lp)+(mp)\,(kn)\,(li)] & -[(pk)\,(nl)\,(im)+(km)\,(pl)\,(in)] & -[(mp)\,(li)\,(nk)+(pk)\,(mi)\,(nl)] \end{vmatrix}$$

which vanishes.

XXVIII. $Z_{iklmnp} + Z_{imlpnk} + Z_{iplknm} \equiv 3\, S_{inl,kmp}$.

Proof: The left-hand-side is

$$3\, S_{inl,kmp} - [\, K_{iklmnp} + K_{imlpnk} + K_{iplknm} \,]$$

and the form between brackets vanishes. q.e.d.

XXIX. $Z_{kmplin} + Z_{pkminl} + Z_{mpknli} \equiv S_{iln,kmp}$.

XXX. The STEINER-point and the KIRKMAN-points form harmonical groups with the points $Z$ and the SALMON-points on a CAYLEY–SALMON-line.

Proof: We have the relations:

$$Z_{iklmnp} = S_{inl,kmp} - K_{iklmnp}$$
$$\Sigma(im, nk, lp) = S_{inl,kmp} + K_{iklmnp}. \qquad \text{q.e.d.}$$

XXXI. $Z_{kmplin}, Z_{pkminl}, Z_{mpknli}$ are collinear on $z_{iklmnp}$.

Proof: We have the relations:

$$Z_{kmplin} = S_{kip,mln} - K_{kmplin} \quad ; \quad \Sigma(im, lk, np) = S_{kip,mln} + K_{kmplin},$$
$$Z_{pkminl} = S_{pnm,kil} - K_{pkminl} \quad ; \quad \Sigma(ip, lm, nk) = S_{pnm,kil} + K_{pkminl},$$
$$Z_{mpknli} = S_{mlk,pni} - K_{mpknli} \quad ; \quad \Sigma(ik, lp, nm) = S_{mlk,pni} + K_{mpknli}.$$

Now the STEINER-points are on the line of STEINER–PLÜCKER $j(im, lp, nk)$, the KIRKMAN-points are on $p_{iklmnp}$ and the SALMON-points on $c_{iln,mkp}$. So the $Z$-points are collinear on a line through $S_{iln,kmp}$ the fourth harmonical of $p_{iklmnp}$ to $j(im, lp, nk)$ and $c_{iln,mkp}$.

XXXII. $z_{iklmnp} \equiv 2\, \langle iklmnp \rangle\, p_{iklmnp} + \langle imlpnk \rangle\, p_{imlpnk} +$
$+ \langle iplknm \rangle\, p_{iplknm}$.

Proof:

$$c_{inl,kmp} \equiv \langle iklmnp \rangle\, p_{iklmnp} + \langle imlpnk \rangle\, p_{imlpnk} + \langle iplknm \rangle\, p_{iplknm};$$

$$-(im)(lp)(nk)\, j(im, lp, nk) \equiv \langle imlpnk \rangle\, p_{imlpnk} + \langle iplknm \rangle\, p_{iplknm}$$
$$\equiv c_{inl,kmp} - \langle iklmnp \rangle\, p_{iklmnp}.$$

So $z_{iklmnp} \equiv c_{inl,kmp} + < iklmnp > p_{iklmnp}$. q.e.d.

We have $z_{iklmnp} + z_{imlpnk} + z_{iplknm} \equiv 4\, c_{inl,kmp}$, as is evident from the relation just deduced.

XXXIII. The linear relation between $z_{kmplin}$, $z_{pkminl}$, $z_{mpknli}$, which are concurrent in $Z_{iklmnp}$ is

$$z_{kmplin} + z_{pkminl} + z_{mpknli} \equiv 0.$$

Proof: $z_{kmplin} = c_{kip,mln} + < kmplin > p_{kmplin}$ etc. gives by addition for the left-hand-side

$$c_{kip,mln} + c_{pnm,kil} + c_{mlk,pni} + c_{iln,mpk} \equiv 0.$$

*(To be continued.)*

**Mathematics.** — *Sequences of points on a circle.* By N. G. DE BRUIJN and P. ERDÖS. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of December 18, 1948.)

## 1. Introduction.

We consider sequences $\{a\}$ of points $a_1, a_2, a_3, \ldots$ on a circle with radius $1/2\pi$, in other words numbers mod 1. The numbers $a_1, a_2, \ldots, a_n$ define $n$ intervals with total length 1; denote by $M_n^1(a)$ and $m_n^1(a)$ the largest and the smallest length. Clearly

$$n\,M_n^1(a) \geqq 1 \geqq n\,m_n^1(a).$$

Analogously $M_n^r(a)$ and $m_n^r(a)$ denote the maximum and minimum length of the sum of $r$ consecutive intervals, so that $n\,M_n^r(a) \geqq r \geqq n\,m_n^r(a)$. We put

$$\limsup_{n\to\infty} n\,M_n^r(a) = \Lambda_r(a)$$

$$\liminf_{n\to\infty} n\,m_n^r(a) = \lambda_r(a)$$

$$\limsup_{n\to\infty} M_n^r(a)/m_n^r(a) = \mu_r(a)$$

and

$$\Lambda_r = \text{g.l.b.}\ \Lambda_r(a) \quad , \quad \lambda_r = \text{l.u.b.}\ \lambda_r(a) \quad , \quad \mu_r = \text{g.l.b.}\ \mu_r(a).$$

We are able to determine

$$\Lambda_1 = 1/\log 2 \quad , \quad \lambda_1 = 1/\log 4 \quad , \quad \mu_1 = 2.$$

The problem of $\Lambda_r$, $\lambda_r$, $\mu_r$ is closely related to a problem concerning "just distributions" solved by Mrs VAN AARDENNE-EHRENFEST [1]. All we can prove is that $\mu_r \geqq 1 + 1/r$ (and analogus inequalities for $\Lambda_r$ and $\lambda_r$); we conjecture that $r(\mu_r - 1)$ is unbounded. From this the theorem of Mrs VAN AARDENNE-EHRENFEST would follow.

## 2. A sequence which gives the best possible values of $\Lambda_1(a)$, $\lambda_1(a)$, $\mu_1(a)$.

Take $a_k = {}^2\log(2k-1)$, reduced mod 1. We show that $a_1, \ldots, a_n$ occur in the following order

$$ {}^2\log n,\ {}^2\log(n+1), \ldots, {}^2\log(2n-1). \quad . \quad . \quad . \quad . \quad (2.1)$$

Namely, no two of the $a_k$'s and no two of the numbers (2.1) are congruent mod 1, but each number in (2.1) is congruent to just one $a_k$.

It follows from (2.1) that the lengths of the intervals defined by $a_1, \ldots, a_n$ are

$$ {}^2\log\frac{n+1}{n},\ {}^2\log\frac{n+2}{n+1}, \ldots, {}^2\log\frac{2n-1}{2n-2},\ {}^2\log\frac{2n}{2n-1},$$

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[1] Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam 48, 266—271 (1945) = Indagationes Mathematicae, 7, 71—76 (1946).

and so

$$n\,M_n^1(a) = \frac{n\log\left(1+\frac{1}{n}\right)}{\log 2}, \quad n\,m_n^1(a)\,\frac{n\log\left(1-\frac{1}{2n}\right)^{-1}}{\log 2}.$$

For $n \to \infty$, $n\,M_n^1(a)$ increases to the limit $1/\log 2$; $n\,m_n^1(a)$ decreases to the limit $1/\log 4$; $M_n^1(a)/m_n^1(a)$ increases to the limit 2. It follows that $\Lambda_1(a) = 1/\log 2$, $\lambda_1(a) = 1/\log 4$, $\mu_1(a) = 2$.

## 3. Lower bound for $\Lambda_r(a)$.

Let $\{a\}$ be a sequence, $n$ a natural number, and suppose that $\varrho$ is such that

$$k\,M_n^1(a) < \varrho. \qquad (n \leqslant k < 2n) \quad . \; . \; . \; . \; . \quad (3.1)$$

Let the intervals determined by $a_1, \ldots, a_n$ be $I_1, \ldots, I_n$, arranged in descending order of length. Denote the length of $I_j$ by $\alpha_j$; so that

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n; \qquad \alpha_1 + \ldots + \alpha_n = 1. \quad . \; . \; . \quad (3.2)$$

Now put in the points $a_{n+1}, a_{n+2}, \ldots, a_{2n-1}$. Since any point "destroys" one $I$ at most, there remains at least one interval of length $\geqslant \alpha_p$ undisturbed after $a_{n+1}, \ldots, a_{n+p-1}$ have been put in $(1 \leqslant p \leqslant n)$. Hence

$$M_n^1(a) \geqslant \alpha_1, \; M_{n+1}^1(a) \geqslant \alpha_2, \ldots, \; M_{2n-1}^1(a) \geqslant \alpha_n;$$

consequently, by (3.1) and (3.2),

$$\varrho\left(\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1}\right) > 1.$$

It follows that for at least one $k\,(n \leqslant k < 2n)$ we have

$$k\,M_k^1(a) \geqslant \left(\frac{1}{n} + \ldots + \frac{1}{2n-1}\right)^{-1} = \sigma_n.$$

We have $\sigma_n < 1/\log 2$, $\sigma_n \to 1/\log 2$, and so $\Lambda_1(a) \geqslant 1/\log 2$. This holds for any $\{a\}$; the lower bound is attained for the sequence of section 2.

Similarly we can prove that for at least one $k\,(rn \leqslant k < (r+1)\,n)$ we have

$$k\,M_k^r(a) \geqslant \left(\frac{1}{rn} + \frac{1}{rn+1} + \ldots + \frac{1}{rn+n-1}\right)^{-1}$$

and so

$$\Lambda_r(a) \geqslant 1/\log\left(1+\frac{1}{r}\right) > r.$$

## 4. Upper bound for $\lambda_r(a)$ 2).

Let $\{a\}$ be a sequence, $n$ a natural number, and suppose that $\varrho$ is such that

$$k\,m_k^1(a) > \varrho \qquad (n < k \leqslant 2n). \quad . \; . \; . \; . \; . \; . \quad (4.1)$$

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2) The proof presented in this section was found by Mrs. VAN AARDENNE-EHRENFEST independently.

Let $a_{k_1}, a_{k_2}, \ldots, a_{k_{2n}}$ be the cyclic order of the points $a_1, \ldots, a_{2n}$ on the circle ($k_1, \ldots, k_{2n}$ is a permutation of $1, \ldots, 2n$); put $k_{2n+1} = k_1$. If $k_i^* = \operatorname{Max}(k_i, k_{i+1}, n+1)$, then the interval $a_{k_i}, a_{k_{i+1}}$ is one of the intervals determined by $a_1, \ldots, a_{k_i^*}$. It follows that its length is less than $\varrho/k_i^*$. Hence

$$1 > \varrho \sum_{i=1}^{2n} 1/k_i^* . \quad \ldots \ldots \ldots \quad (4.2)$$

We have $n < k_i^* \leqslant 2n$, and any $k\,(n+1 < k \leqslant 2n)$ occurs $\varepsilon_k$ times as a $k^*$; $\varepsilon_k = 0, 1$ or $2$. It follows that

$$\sum_{i=1}^{2n} 1/k_i^* = \sum_{n+1}^{2n} \frac{2}{k} + \sum_{n+2}^{2n} (2-\varepsilon_k) \left\{ \frac{1}{n+1} - \frac{1}{k} \right\} \geqslant \sum_{n+1}^{2n} \frac{2}{k} .$$

Finally, by (4.1) and (4.2) we infer that at least for one ($n < k \leqslant 2n$) we have

$$k\, m_k^1(\mathrm{a}) \leqslant \left( \frac{2}{n+1} + \frac{2}{n+2} + \ldots + \frac{2}{2n} \right)^{-1} = \tau_n .$$

We have $\tau_n > 1/\log 4$, $\tau_n \to 1/\log 4$, and so $\lambda_1(\mathrm{a}) \leqslant 1/\log 4$. The example of section 2 again shows that $1/\log 4$ is best possible.

Similarly we can show that for at least one $k\,(rn < k \leqslant (r+1)\,n)$ we have

$$k\, m_k^r(\mathrm{a}) \leqslant r \left( \frac{r+1}{nr+1} + \ldots + \frac{r+1}{nr+n-1} \right)^{-1}$$

and so

$$\lambda_r(\mathrm{a}) \leqslant \frac{r}{r+1} \Big/ \log\left(1+\frac{1}{r}\right) < r . \quad \ldots \ldots \quad (4.3)$$

## 5. Lower bound for $\mu_r$.

Let $\{a\}$ be a sequence. We first prove that, for $r \geqslant 1$, $n \geqslant 1$ we have

$$M_n^r(\mathrm{a}) / m_{n+1}^r(\mathrm{a}) \geqslant 1 + \frac{1}{r} . \quad \ldots \ldots \ldots \quad (5.1)$$

We first suppose that $r > 1$. Let $I_1, I_2, \ldots, I_n$ be the intervals of the $n$-th stage, i.e. the intervals determined by $a_1, \ldots, a_n$. Let $I_{k_0}$ be the one into which $a_{n+1}$ falls, and let

$$I_{k_{-r+1}}, I_{k_{-r+2}}, \ldots, I_{k_0}, I_{k_1}, \ldots, I_{k_{r-1}} \quad \ldots \ldots \quad (5.2)$$

be consecutive on the circle [3]).

Put $M = M_n^r(\mathrm{a})$, $m = m_{n+1}^r(\mathrm{a})$ and denote by $M_1$ the maximum length of the sum of $r$ consecutive intervals from the set (5.2). Denote the length of $I_{k_i}$ by $\beta_i$. Let $\gamma_1$ and $\gamma_2$ be the lengths of the parts into which $I_{k_0}$ is divided by $a_{n+1}$.

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[3]) If $2r-1 > n$ the $k_i$ are not all different.

Clearly at least one of the numbers $\beta_{-r+1},\ldots,\beta_{-1},\beta_1,\ldots,\beta_{r-1}$ ($\beta_j$ say) is $\geqslant (M_1-\beta_0)/(r-1)$; we may suppose that $j>0$. Now we have

$$m\leqslant \beta_{j-r+1}+\ldots+\beta_{-1}+\gamma_1+\gamma_2+\ldots+\beta_{j-1}$$

and hence

$$m\leqslant M_1-\beta_j\leqslant \frac{r-2}{r-1}M_1+\frac{\beta_0}{r-1}. \quad . \quad . \quad . \quad . \quad (5.3)$$

On the other hand it follows from

$$m\leqslant \gamma_2+\beta_1+\ldots+\beta_{r-1}\leqslant M_1-\gamma_1$$

$$m\leqslant \beta_{-r+1}+\ldots+\beta_{-1}+\gamma_1\leqslant M_1-\gamma_2$$

that

$$m\leqslant M_1-\tfrac{1}{2}\beta_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

Trivially we have $M_1\leqslant M$. If $\beta_0\leqslant 2M_1/(r+1)$ we infer $m\leqslant M_1 r/(1+r)\leqslant Mr/(1+r)$ from (5.3); if $\beta_0\geqslant 2M_1/(r+1)$ we deduce the same result from (5.4). This proves (5.1) for $r>1$.

If $r=1$, (5.1) immediately follows from

$$m\leqslant \operatorname{Min}(\gamma_1,\gamma_2)\leqslant \tfrac{1}{2}\beta_0\leqslant \tfrac{1}{2}M.$$

Now suppose that $n$ is a natural number and that for $nr\leqslant k\leqslant n(r+1)$ we have

$$M_k^r(a)/m_k^r(a)<\left(1+\frac{1}{r}\right)\Big/\left(1+\frac{1}{k}\right)^2. \quad . \quad . \quad . \quad . \quad (5.5)$$

It follows, by (5.1) that

$$m_{k+1}^r/m_k^r<k^2/(k+1)^2 \qquad (nr\leqslant k<nr+n)$$

and also

$$m_{rn+n}^r/m_{rn}^r<r^2/(1+r)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.6)$$

Trivially we have $m_{rn}^r\leqslant 1/n$; on the other hand, by (5.5)

$$m_{rn+n}^r>\frac{r}{1+r}M_{rn+n}^r\geqslant\frac{r}{1+r}\cdot\frac{r}{rn+n-1}\geqslant\frac{r^2}{(r+1)^2}\cdot\frac{1}{n}.$$

This contradicts (5.6). Hence for at least one $k\,(nr\leqslant k\leqslant nr+n)$ (5.5) is not true. It follows that

$$\mu_r\geqslant 1+\frac{1}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.7)$$

6. The inequalities (3.3), (4.3) and (5.7) are probably not best possible if $r\geqslant 2$. We conjecture that the expressions

$$r(\Lambda_r-1) \quad , \quad r(1-\lambda_r) \quad , \quad r(\mu_r-1)$$

tend to infinity if $r\to\infty$.

We owe some useful remarks to Mrs. T. VAN AARDENNE-EHRENFEST and Mr. J. KOREVAAR with whom we first discussed the above problems.

**Mathematics.** — *Lattice points in non-convex regions.* III. By P. MULLENDER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 27, 1948.)

**4. The Proof.** Suppose $P_1 = (x_1', \dots, x_n')$ and $P_2 = (x_1'', \dots, x_n'')$ are points of $\overline{K}$. Then we have to prove that $P_1 + P_2$ is a point of $\overline{R}$, i.e.

$$F(x_1' + x_1'', \dots, x_n' + x_n'') \leqq F(1, \dots, 1). \quad . \quad . \quad . \quad . \quad (86)$$

We can divide the coordinates $x_\nu$ into four groups, according as

a. Both $x_\nu'$ and $x_\nu''$ are not greater than 1.
b. Only $x_\nu''$ is greater than 1, but not $x_\nu'$.
c. Only $x_\nu'$ is greater than 1, but not $x_\nu''$.
d. Both $x_\nu'$ and $x_\nu''$ are greater than 1.

Of course it is possible that some of these groups do not contain any members at all. Without loss of generality we may suppose that the first $p$ coordinates belong to group a, the next $q$ to group b, the next $r$ to group c and the last $s$ to group d, where $p + q + r + s = n$.

Now it is convenient to choose our notation in accordance with the division of the coordinates into these four groups. For, in our proof, different coordinates of the same group have to be dealt with in the same way, whereas coordinates of different groups have to be dealt with in different ways. Hence we need not deal with all the coordinates, one by one, but we may take one representative for each group.

We denote the coordinates of group a by $a_1, \dots, a_p$, those of group b by $b_1, \dots, b_q$, etc. Then, writting $a$ for $a_1, \dots, a_p$, etc., we have

$$F(x_1, \dots, x_n) \equiv F(a, b, c, d),$$

and in particular

$$F(1, \dots, 1) = F(1, 1, 1, 1).$$

Also ($\alpha = 1, \dots, p$ and $\beta = 1, \dots, q$ etc.)

$$\frac{\partial}{\partial x_\alpha} F(x_1, \dots, x_n) \equiv \frac{\partial}{\partial a_\alpha} F(a, b, c, d)$$

and

$$\frac{\partial}{\partial x_{p+\beta}} F(x_1, \dots, x_n) \equiv \frac{\partial}{\partial b_\beta} F(a, b, c, d), \text{ etc.}$$

We use, further, the following abbreviations:

$$\sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1) = A \ , \quad \sum_{\beta=1}^{q} \frac{\partial}{\partial b_\beta} F(1, 1, 1, 1) = B \text{ etc.} \quad . \quad (87)$$

Now we have

$$P_1=(a',b',c',d') \quad , \quad P_2=(a'',b'',c'',d''), \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

with

$$a'\leqq 1, \; b'\leqq 1, \quad a''\leqq 1, \; c''\leqq 1, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (89)$$

and

$$c'>1, \; d'>1, \; b''>1, \; d''>1 \; . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (90)$$

That means $P_1$ is a point of $\overline{K}(1, \ldots, p, p+1, \ldots, p+q)$ and $P_2$ is a point of $\overline{K}(1, \ldots, p, p+q+1, \ldots, p+q+r)$.

By (73),

$$\vartheta'=\frac{A+B+C+D}{2(A+B)}, \qquad \vartheta''=\frac{A+B+C+D}{2(A+C)}, \quad . \quad . \quad . \qquad (91)$$

and, by (74),

$$\left.\begin{aligned} x' &= \frac{\sum\limits_{\alpha=1}^{p} a'_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum\limits_{\beta=1}^{q} b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1)}{A+B}, \\ x'' &= \frac{\sum\limits_{\alpha=1}^{p} a''_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum\limits_{\gamma=1}^{r} c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1)}{A+C}. \end{aligned}\right\} \quad . \quad . \qquad (92)$$

Finally, by (77),

$$\left.\begin{aligned} F(x'+\vartheta', x'+\vartheta', c', d') &\leqq F(1,1,1,1), \\ F(x''+\vartheta'', b'', x''+\vartheta'', d'') &\leqq F(1,1,1,1). \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (93)$$

We have to prove (86), or in our new notation,

$$F(a'+a'', b'+b'', c'+c'', d'+d'')\leqq F(1,1,1,1). \quad . \quad . \qquad (94)$$

We do this in four steps.

(i) First we assert that the group of coordinates a cannot be empty. For, if that were the case, we should have $A=0$, and hence, by (91),

$$\vartheta'+\vartheta''=\frac{B+C+D}{2B}+\frac{B+C+D}{2C}\geqq\frac{(B+C)^2}{2BC}\geqq 2.$$

This cannot be true, since $\vartheta'<1$ and $\vartheta''<1$, because the regions to which $P_1$ and $P_2$ belong cannot be empty.

We now define $y'$ and $y''$ by

$$F(y'+\vartheta', y'+\vartheta', c', d')=F(1,1,1,1)$$

and

$$\left.\begin{aligned} F(y''+\vartheta'', b'', y''+\vartheta'', d'')=F(1,1,1,1) \end{aligned}\right\} \quad . \quad . \quad . \qquad (95)$$

respectively. Note that $y'$ and $y''$ each denote one number only, whereas $c'$, $d'$, $b''$ and $d''$ each denote a set of numbers, namely $c'_1, \ldots, c'_r$ etc. Then, by (93) and (63),

$$y'\geqq x' \quad \text{and} \quad y''\geqq x'',$$

or, by (92),

$$\left.\begin{aligned} y' &\geqq \frac{\sum_{\alpha=1}^{p} a'_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum_{\beta=1}^{q} b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1)}{A+B}, \\ y'' &\geqq \frac{\sum_{\alpha=1}^{p} a''_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) + \sum_{\gamma=1}^{r} c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1)}{A+C}. \end{aligned}\right\} \quad . \quad . \quad (96)$$

This we may write in the form

$$\sum_{\alpha=1}^{p} a'_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) - y'A + \sum_{\beta=1}^{q} b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - y'B \leqq 0,$$

$$\sum_{\alpha=1}^{p} a''_\alpha \frac{\partial}{\partial a_\alpha} F(1,1,1,1) - y''A + \sum_{\gamma=1}^{r} c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1) - y''C \leqq 0,$$

or, by (87),

$$\sum_{\alpha=1}^{p} \left\{ a'_\alpha - y' + \frac{\sum_{\beta=1}^{q} b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - y'B}{A} \right\} \frac{\partial}{\partial a_\alpha} F(1,1,1,1) \leqq 0,$$

$$\sum_{\alpha=1}^{p} \left\{ a''_\alpha - y'' + \frac{\sum_{\gamma=1}^{r} c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1) - y''C}{A} \right\} \frac{\partial}{\partial a_\alpha} F(1,1,1,1) \leqq 0.$$

Adding we obtain

$$\sum_{\alpha=1}^{p} (a'_\alpha + a''_\alpha - X) \frac{\partial}{\partial a_\alpha} F(1,\,1,\,1,\,1) \leqq 0, \quad . \quad . \quad . \quad . \quad (97)$$

where

$$X = y' + y'' + \frac{y'B + y''C - \sum_{\beta=1}^{q} b'_\beta \frac{\partial}{\partial b_\beta} F(1,1,1,1) - \sum_{\gamma=1}^{r} c''_\gamma \frac{\partial}{\partial c_\gamma} F(1,1,1,1)}{A}. \quad (98)$$

However, for $\alpha_1, \alpha_2 = 1, \ldots, p$,

$$\frac{\frac{\partial}{\partial a_{\alpha_1}} F(1,1,1,1)}{\frac{\partial}{\partial a_{\alpha_2}} F(1,1,1,1)} = \frac{\frac{\partial}{\partial a_{\alpha_1}} F(X,\, b' + b'',\, c' + c'',\, d' + d'')}{\frac{\partial}{\partial a_{\alpha_2}} F(X,\, b' + b'',\, c' + c'',\, d' + d'')}.$$

Hence (97) implies

$$\sum_{\alpha=1}^{p} (a'_\alpha + a''_\alpha - X) \frac{\partial}{\partial a_\alpha} F(X,\, b' + b'',\, c' + c'',\, d' + d'') \leqq 0.$$

From this it follows, by lemma 1, that

$$F(a' + a'',\, b' + b'',\, c' + c'',\, d' + d'') \leqq F(X,\, b' + b'',\, c' + c'',\, d' + d''). \quad (99)$$

(ii) We now prove that

$$F(X,\, b' + b'',\, c' + c'',\, d' + d'')$$

does not decrease, if one of the coordinates $b'$ or $c''$ decreases, suppose $b'_1$. We leave out this step, if both the groups $b$ and $c$ are empty.

By (98),

$$\frac{dX}{db'_1} = \frac{-\frac{\partial}{\partial b_1} F(1, 1, 1, 1)}{A},$$

since, by (95), $y'$ and $y''$ depend only on the coordinates $c'$, $d'$, $b''$ and $d''$. Hence

$$\frac{d}{db'_1} F(X, b' + b'', c' + c''; d' + d'') =$$

$$= \frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') +$$

$$+ \sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'') \cdot \frac{dX}{db'_1} =$$

$$= \frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') -$$

$$- \frac{\frac{\partial}{\partial b_1} F(1, 1, 1, 1) \sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'')}{A}.$$

Now we have to prove that this expression is not positive.

Multiplication by

$$\frac{A}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') \frac{\partial}{\partial b_1} F(1, 1, 1, 1)} =$$

$$= \frac{\sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1)}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'') \frac{\partial}{\partial b_1} F(1, 1, 1, 1)}$$

does not alter the sign, since, by (63), none of the derivatives of is negative. Thus we obtain

$$\sum_{\alpha=1}^{p} \left\{ \frac{\frac{\partial}{\partial a_\alpha} F(1, 1, 1, 1)}{\frac{\partial}{\partial b_1} F(1, 1, 1, 1)} - \frac{\frac{\partial}{\partial a_\alpha} F(X, b' + b'', c' + c'', d' + d'')}{\frac{\partial}{\partial b_1} F(X, b' + b'', c' + c'', d' + d'')} \right\}.$$

According to our assumption that

$$\frac{\frac{\partial}{\partial x_k} F(x_1, \ldots, x_n)}{\frac{\partial}{\partial x_l} F(x_1, \ldots, x_n)}$$

increases, if $x_k/x_l$ decreases, no term of the above sum will be positive, if

$$\frac{X}{b_1'+b_1''}\leqq 1. \qquad \dots\dots\dots\dots \qquad (100)$$

By (98),

$$X\leqq y'+y''+\frac{y'B+y''C}{A}=\frac{y'(A+B)+y''(A+C)}{A}.$$

Further, by (63), (90) and (95),

$$y'+\vartheta'\leqq 1 \quad , \quad y''+\vartheta''\leqq 1.$$

Hence, by (91),

$$y'\leqq\frac{A+B-C-D}{2(A+B)}, \quad y''\leqq\frac{A+C-B-D}{2(A+C)},$$

and so

$$X\leqq\frac{A+B-C-D+A+C-B-D}{2A}=\frac{A-D}{A}\leqq 1.$$

However, by (90), $b_1''>1$, and hence indeed (100) is true.

Thus we have proved that

$$F(X,b'+b'',c'+c'',d'+d'')$$

attains its largest value, when the coordinates $b'$ and $c''$ are least, i.e. when $b_1'=\dots=b_q'=c_1''=\dots=c_r''=0$. Hence

$$F(X,b'+b'',c'+c'',d'+d'')\leqq F(X_1,b'',c',d'+d''), \quad . \quad (101)$$

where

$$X_1=\frac{y'(A+B)+y''(A+C)}{A}. \qquad \dots\dots \qquad (102)$$

(iii) We prove, further, that

$$F(X_1,b'',c',d'+d'')$$

does not decrease, if one of the coordinates $b''$ or $c'$ decreases, suppose $c_1'$. We also leave out this step, if there are no coordinates $b$ and $c$.

By (102),

$$\frac{dX_1}{dc_1'}=\frac{A+B}{A}\cdot\frac{dy'}{dc_1'},$$

but, for $\alpha=1,\dots,p$; $\beta=1,\dots,q$,

$$\frac{\dfrac{\partial}{\partial a_\alpha}F(1,1,1,1)}{\dfrac{\partial}{\partial b_\beta}F(1,1,1,1)}=\frac{\dfrac{\partial}{\partial a_\alpha}F(y'+\vartheta',y'+\vartheta',c',d')}{\dfrac{\partial}{\partial b_\beta}F(y'+\vartheta',y'+\vartheta',c',d')},$$

and so

$$\frac{dX_1}{dc_1'}=\frac{\sum\limits_{\alpha=1}^{p}\dfrac{\partial}{\partial a_\alpha}F(y'+\vartheta',y'+\vartheta',c',d')+\sum\limits_{\beta=1}^{q}\dfrac{\partial}{\partial b_\beta}F(y'+\vartheta',y'+\vartheta',c',d')}{\sum\limits_{\alpha=1}^{p}\dfrac{\partial}{\partial a_\alpha}F(y'+\vartheta',y'+\vartheta',c',d')}\cdot\frac{dy'}{dc_1'}.$$

Further, by (95),

$$\frac{d}{dc_1'} F(y' + \vartheta', y' + \vartheta', c', d') = 0,$$

or

$$\frac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d') + \left\{ \sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d') + \right.$$

$$\left. + \sum_{\beta=1}^{q} \frac{\partial}{\partial b_\beta} F(y' + \vartheta', y' + \vartheta', c', d') \right\} \frac{dy'}{dc_1'} = 0,$$

whence it follows that

$$\frac{dy'}{dc_1'} = - \frac{\dfrac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y'+\vartheta', y'+\vartheta', c', d') + \sum\limits_{\beta=1}^{q} \dfrac{\partial}{\partial b_\beta} F(y'+\vartheta', y'+\vartheta', c', d')}.$$

Substitution gives

$$\frac{dX_1}{dc_1'} = - \frac{\dfrac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}.$$

Hence

$$\frac{d}{dc_1'} F(X_1, b'', c', d' + d'') =$$

$$= \frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') + \sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(X_1, b'', c', d'+d'') \cdot \frac{dX_1}{dc_1'} =$$

$$= \frac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') -$$

$$- \frac{\dfrac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d') \sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(X_1, b'', c', d' + d'')}{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}.$$

Again we have to prove that this expression cannot be positive. Multiplication by

$$\frac{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta, c', d')}{\dfrac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'') \dfrac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')}$$

does not alter the sign. Thus we obtain

$$\sum_{\alpha=1}^{p} \left\{ \frac{\dfrac{\partial}{\partial a_\alpha} F(y' + \vartheta', y' + \vartheta', c', d')}{\dfrac{\partial}{\partial c_1} F(y' + \vartheta', y' + \vartheta', c', d')} - \frac{\dfrac{\partial}{\partial a_\alpha} F(X_1, b'', c', d' + d'')}{\dfrac{\partial}{\partial c_1} F(X_1, b'', c', d' + d'')} \right\},$$

and according to our assumption that

$$\frac{\dfrac{\partial}{\partial x_k} F(x_1, \ldots, x_n)}{\dfrac{\partial}{\partial x_l} F(x_1, \ldots, x_n)}$$

increases, if $x_k/x_l$ decreases, no term of the above sum will be positive, if

$$\frac{X_1}{c_1'} \leqq \frac{y' + \vartheta'}{c_1'}$$

or

$$X_1 \leqq y' + \vartheta',$$

or, by (102),

$$y'(A + B) + y''(A + C) \leqq (y' + \vartheta')A,$$

i.e.

$$y'B + y''(A + C) - \vartheta'A \leqq 0.$$

As before we have

$$y' + \vartheta' \leqq 1, \quad y'' + \vartheta'' \leqq 1,$$

hence

$$y'B + y''(A + C) - \vartheta'A \leqq B + A + C - (A + B)\vartheta' - (A + C)\vartheta'',$$

or, by (91),

$$y'B + y''(A + C) - \vartheta'A \leqq B + A + C - \\ - 2 \cdot \tfrac{1}{2}(A + B + C + D) = -D \leqq 0.$$

This proves our assertion. Hence

$$F(X_1, b'', c', d' + d'')$$

attains its largest value, when the coordinates $b''$ and $c'$ are least, i.e., by (90), when $b_1'' = \ldots = b_q'' = c_1' = \ldots = c_r' = 1$. Consequently

$$F(X_1, b'', c', d' + d'') \leqq F(X_2, 1, 1, d' + d''), \qquad (103)$$

where

$$X_2 = \frac{y'(A + B) + y''(A + C)}{A}, \qquad (104)$$

with $y'$ and $y''$ given by (95) under the assumption that

$$b_1'' = \ldots = b_q'' = c_1' = \ldots = c_r' = 1.$$

(iv) We finally prove that

$$F(X_2, 1, 1, d' + d'')$$

does not decrease, if one of the coordinates $d'$ or $d''$ decreases, suppose $d_1'$. We leave out this step, if there are no coordinates $d$.

By (104),

$$\frac{dX_2}{dd_1'} = \frac{A + B}{A} \cdot \frac{dy'}{dd_1'}.$$

From this it follows in the same way as before, that

$$\frac{dX_2}{dd_1'} = - \frac{\dfrac{\partial}{\partial d_1} F(y'+\vartheta', y'+\vartheta', c', d')}{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y'+\vartheta', y'+\vartheta', c', d')}.$$

Hence

$$\frac{d}{dd_1'} F(X_2, 1, 1, d'+d'') =$$

$$= \frac{\partial}{\partial d_1} F(X_2, 1, 1, d'+d'') + \sum_{\alpha=1}^{p} \frac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d'+d'') \frac{dX_2}{dd_1'} =$$

$$= \frac{\partial}{\partial d_1} F(X_2, 1, 1, d'+d'') -$$

$$- \frac{\dfrac{\partial}{\partial d_1} F(y'+\vartheta', y'+\vartheta', c', d') \sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d'+d'')}{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y'+\vartheta', y'+\vartheta', c', d')}.$$

Multiplication by

$$\frac{\sum\limits_{\alpha=1}^{p} \dfrac{\partial}{\partial a_\alpha} F(y'+\vartheta', y'+\vartheta', c', d')}{\dfrac{\partial}{\partial d_1} F(X_2, 1, 1, d'+d'') \cdot \dfrac{\partial}{\partial d_1} F(y'+\vartheta', y'+\vartheta', c', d')}$$

does not alter the sign. Thus we obtain

$$\sum_{\alpha=1}^{p} \left\{ \frac{\dfrac{\partial}{\partial a_\alpha} F(y'+\vartheta', y'+\vartheta', c', d')}{\dfrac{\partial}{\partial d_1} F(y'+\vartheta', y'+\vartheta', c', d')} - \frac{\dfrac{\partial}{\partial a_\alpha} F(X_2, 1, 1, d'+d'')}{\dfrac{\partial}{\partial d_1} F(X_2, 1, 1, d'+d'')} \right\}.$$

This is not positive, if

$$\frac{X_2}{d_1' + d_1''} \leqq \frac{y'+\vartheta'}{d_1'},$$

which is true, since $X_2$ is a special value of $X_1$, and $X_1 \leqq y' + \vartheta'$, which we have proved already.

This proves the assertion, and hence

$$F(X_2, 1, 1, d'+d'')$$

attains its largest value, when $d_1' = \ldots = d_s' = d_1'' = \ldots = d_s'' = 1$. So we have finally

$$F(X_2, 1, 1, d'+d'') \leqq F(X_3, 1, 1, 2), \quad \ldots \ldots \quad (105)$$

where again

$$X_3 = \frac{y'(A+B) + y''(A+C)}{A}, \quad \ldots \ldots \quad (106)$$

with $y'$ and $y''$ given by (95), now, however, under the assumption, that not only $b_1'' = \ldots = b_q'' = c_1' = \ldots = c_r' = 1$, but also

$$d_1' = \ldots = d_s' = d_1'' = \ldots = d_s'' = 1.$$

Under this assumption we have, by (95),

$$y' + \vartheta' = 1 \quad \text{and} \quad y'' + \vartheta'' = 1,$$

so that, by (91),

$$y' = \frac{A+B-C-D}{2(A+B)} \quad \text{and} \quad y'' = \frac{A+C-B-D}{2(A+C)}.$$

Substituting these values in (106) we get

$$X_3 = \frac{A+B-C-D+A+C-B-D}{2A} = \frac{A-D}{A}.$$

By (99), (101), (103) and (105), we have now

$$F(a'+a'', b'+b'', c'+c'', d'+d'') \leqq F(X_3, 1, 1, 2).$$

If some of the four groups of coordinates are empty, then the corresponding terms in this inequality have to be left out. E.g. if there are no coordinates $b$, $c$ and $d$, i.e. if only group a is not empty, we have

$$F(a'+a'') \leqq F(X_3),$$

where $X_3 = X = 1$.

Now, by lemma 1,

$$F(X_3, 1, 1, 2) - F(1, 1, 1, 1) \leqq \left(\frac{A-D}{A} - 1\right) A + (2-1) D = 0,$$

and hence also

$$F(a'+a'', b'+b'', c'+c'', d'+d'') \leqq F(1, 1, 1, 1),$$

which proves the theorem.

**5. The Application.** If we define

$$F(x_1, \ldots, x_n) \equiv (x_1 \ldots x_n)^{\frac{1}{n}},$$

then $F(x_1, \ldots, x_n)$ satisfies all the conditions mentioned in $B$, 1, as one can easily verify. Thus we get the solution of the problem of the product of $n$ homogeneous linear forms we gave in $A$.

It is possible to obtain other solutions as well.

If we put

$$F(X, Y, Z) \equiv (X^p Y^q Z^r)^{\frac{1}{p+q+r}}, \quad \ldots \ldots \quad (107)$$

where

$$X = \frac{1}{p} \sum_{\lambda=1}^{p} |x_\lambda|, \quad Y = \frac{1}{q} \sum_{\mu=1}^{q} |x_{p+\mu}|, \quad Z = \frac{1}{r} \sum_{\nu=1}^{r} |x_{p+q+\nu}|, \quad (108)$$

with $p + q + r = n$, then the region $R$ is three dimensional in the coordinates $X$, $Y$ and $Z$, but, by (108), if we take $p > \frac{1}{2}n$, it represents the $n$-dimensional region, considered by MORDELL, in order to apply his $n$-dimensional method to the product of the $n$ homogeneous linear forms, i.e. the region we denoted in $A$, 2 by $R''$.

Now, our method furnishes a construction of a three dimensional region $K$, for which theorem 6 holds with respect to the three dimensional region $R$. However, by (108), this region $K$ also represents an $n$-dimensional region, and it is very easy to see that also for this region theorem 6 holds, but with respect to $R''$.

In this case $\overline{K}$ consists of four parts, namely $\overline{K}^*(1,2,3)$, $\overline{K}^*(1)$, $\overline{K}^*(1,2)$ and $\overline{K}^*(1,3)$.

By (107) and according to the definition in $B$, 2 the region $\overline{K}^*(1,2,3)$ is given by

$$x + \vartheta \leqq 1. \qquad (109)$$

Since for this region $\vartheta = \frac{1}{2}$ and

$$x = \frac{pX + qY + rZ}{p+q+r},$$

by (68) and (69), we can write (109) in the form

$$pX + qY + rZ \leqq \tfrac{1}{2}n.$$

Therefore, by

$$X = \frac{1}{p}\sum_{\lambda=1}^{p} x_\lambda, \quad Y = \frac{1}{q}\sum_{\mu=1}^{q} x_{p+\mu}, \quad Z = \frac{1}{r}\sum_{\nu=1}^{r} x_{p+q+\nu} \qquad (108a)$$

restricting ourselves again to the first hyperoctant, $\overline{K}^*(1,2,3)$ represents the $n$-dimensional region given by

$$x_1 + \ldots + x_n \leqq \tfrac{1}{2}n, \qquad (110)$$

which region we denoted in $A$, 2 by $\overline{K}_0$.

Further, $\overline{K}^*(1)$ is given by

$$\left.\begin{array}{l} Y > 1,\ Z > 1, \\ \left(X + \dfrac{n}{2p}\right)^p Y^q Z^r \leqq 1, \end{array}\right\} \qquad (111)$$

$\overline{K}^*(1,2)$ by

$$\left.\begin{array}{l} Z > 1, \\ \left(\dfrac{pX + qY + \frac{1}{2}n}{p+q}\right)^{p+q} \cdot Z^r \leqq 1. \end{array}\right\} \qquad (112)$$

and $\overline{K}^*(1,3)$ by

$$\left.\begin{array}{l} Y > 1, \\ \left(\dfrac{pX + rZ + \frac{1}{2}n}{p+r}\right)^{p+r} \cdot Y^q \leqq 1. \end{array}\right\} \qquad (113)$$

The inequalities (111) are the same as we used in $A$, 2 to define $\overline{H}_2$.

Hence $\overline{K}^*(1)$ represents $\overline{H}_2$, plus a set of points, which are also contained in $\overline{K}^*(1, 2, 3) = \overline{K}_0$.

Hence MORDELL's region $\overline{K}$, which we denoted in $A, 2$ by $\overline{K}_2$, and which consisted of $\overline{K}_0$ and $\overline{H}_2$, is wholly contained in the region $\overline{K}$ we can construct by our method.

Moreover we may leave out the restriction $p > \frac{1}{2}n$. The only difference in the result is that we have to replace some the above mentioned four regions by their complementary ones, e.g. $\overline{K}^*(1)$ by $\overline{K}^*(2, 3)$.

To calculate $V_K$ it is more convenient to regard $\overline{K}$ as the sum of the non-overlapping regions $\overline{K}(1, 2, 3)$, $\overline{K}(1, 2)$, $\overline{K}(1, 3)$ and $\overline{K}(1)$. However, it does not seem worth while carrying out the calculation, as our first solution probably gives a much better result.

Generalising these results we can apply our method, to a region $R$ defined by the function

$$F(X_1, \dots, X_N) \equiv (Y_1^{p_1} \dots X_N^{p_N})^{\frac{1}{p_1 + \dots + p_N}},$$

where e.g.

$$X_1 = \frac{1}{p_1} \sum_{\lambda_1=1}^{p_1} |x_\lambda|, \dots, X_N = \frac{1}{p_N} \sum_{\lambda_N=1}^{p_N} |x_{p_1 + \dots p_{N-1} + \lambda_N}|,$$

with $p_1 + \dots + p_N = n$.

Another application of our method can be given by putting

$$F(x_1, \dots, x_n) \equiv (x_1^\sigma + \dots + x_n^\sigma)^{\frac{1}{\sigma}} \qquad (0 < \sigma < 1).$$

Then the region $\overline{R}$ is defined by

$$x_1^\sigma + \dots + x_n^\sigma \leqq n.$$

with $x_1 \geqq 0, \dots, x_n \geqq 0$.

In this case we have for the region $\overline{K}(k_1, \dots, k_p)$,

$$\vartheta = \frac{n}{2p} \quad , \quad x = \frac{1}{p} (x_{k_1}, \dots, k_p),$$

and hence $\overline{K}(k_1, \dots, k_p)$ is defined by the inequalities

$$x_{k_1} \leqq 1, \dots, x_{k_p} \leqq 1,$$
$$x_{l_1} > 1, \dots, x_{l_q} > 1,$$
$$p \left( \frac{x_{k_1} + \dots + x_{k_p} + \frac{1}{2} n}{p} \right)^\sigma + x_{l_1}^\sigma + \dots + x_{l_q}^\sigma \leqq n.$$

The region is not empty, if $\vartheta < 1$, i.e. if $p > \frac{1}{2} n$.

Adding all the regions $\overline{K}(k_1, \dots, k_p)$, which are not empty, we obtain $\overline{K}$.

In conclusion I should like to thank Professor MORDELL for reading the manuscripts of this paper and the previous one under the same title, and for making several useful suggestions.

*St. John's College, Cambridge.*

**Mathematics.** — *On multivectors in a $V_n$.* II. By WŁODZIMIERZ WRONA.
(Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of December 18, 1948.)

Consider now an arbitrary simple non-singular $m$-vector $F^{\lambda_1 \dots \lambda_m}$ and its dual ${}^{\circ}F^{\lambda_{m+1} \dots \lambda_n}$ and denote by $\underset{(m)}{\varkappa}$ the scalar curvature of the $m$-direction which corresponds to $F^{\lambda_1 \dots \lambda_m}$ and by $\underset{(n-m)}{{}^{\circ}\varkappa}$ the scalar curvature of the $(n-m)$-direction absolutely perpendicular to the former one which corresponds to the $(n-m)$-direction of ${}^{\circ}F^{\lambda_{m+1} \dots \lambda_n}$. By $(2.21)$ the relation $(2.62)$ takes now the form

$$\binom{m}{2} \underset{(m)}{\varkappa} - \binom{n-m}{2} \underset{(n-m)}{{}^{\circ}\varkappa} = \tfrac{1}{2}(2m-n)(n-1)\varkappa \quad . \quad . \quad . \quad (2.67)$$

As by $(2.46)$ equation $(2.65)$ may be put in the form

$$W_{[\lambda_1 \lambda_2 [\mu_1 \mu_2} a_{\lambda_3 \mu_3} \dots a_{\lambda_m] \mu_m]} F^{\lambda_1 \dots \lambda_m} F^{\mu_1 \dots \mu_m} = 0,$$

where

$$W_{\lambda_1 \lambda_2 \mu_1 \mu_2} = \varrho^2 m!\,(m\, U_{[\lambda_1 [\mu_1} a_{\lambda_2] \mu_2]} + \mu\, a_{[\lambda_1 [\mu_1} a_{\lambda_2] \mu_2]}),$$

using the lemma I we can give to the last theorem the following equivalent form:

*A necessary and sufficient condition that the space be* EINSTEIN*ian is that the expression*

$$\binom{m}{2} \underset{(m)}{\varkappa} - \binom{n-m}{2} \underset{(n-m)}{{}^{\circ}\varkappa}$$

*be at every point independent of the choice of absolutely perpendicular non-singular m- and (n — m)-directions.*

It has then the value given by $(2.67)$.

Similarly from $(2.60)$ we have:

*A necessary and sufficient condition that the space be conformal to a flat space is that the expression*

$$(n-m) \underset{(n-m)}{{}^{\circ}\omega} + m \underset{(m)}{\omega}$$

*be independent of the choice of the non-singular m-vector at each point of the space.* We have then

$$(n-m) \underset{(n-m)}{{}^{\circ}\omega} + m \underset{(m)}{\omega} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.68)$$

from which in the case when the multivectors under consideration are simple we get by $(2.21)$

$$m \underset{(m)}{\varkappa} + (n-m) \underset{(n-m)}{{}^{\circ}\varkappa} = n\varkappa \quad . \quad . \quad . \quad . \quad . \quad (2.69)$$

and the theorem:

*A necessary and sufficient condition that the space be conformal to a flat space is that the expression*

$$m \underset{(m)}{\varkappa} + (n-m) \underset{(n-m)}{{}^{\circ}\varkappa}$$

*be independent of the choice of absolutely perpendicular non-singular m- and (n — m)-directions at each point of the space.* It has then the value given by (*2.69*).

Consider the subspace $V_m$, $4 \leq m \leq n$, geodesic at a given point of $V_n$ and tangent at this point to the given $m$-direction and denote by

$$'a_{\varkappa\lambda},\ 'K_{\varkappa\lambda\mu\nu},\ 'U_{\varkappa\lambda\mu\nu},\ 'U_{\lambda\mu}$$

and $\underset{(m)}{\varkappa}$ the fundamental tensor, the RIEMANN curvature affinor, the affinors defined by (*2.1*) and (*2.2*) and the scalar curvature of this $V_m$ at the point under consideration. We get the following corollary from the last theorem:

*If the expression*

$$m_1 \underset{(m_1)}{\varkappa} + (m-m_1) \underset{(m-m_1)}{\varkappa}$$

*where* $2 \leq m_1 \leq m-2$ *and* $\underset{(m_1)}{\varkappa}$ *and* $\underset{(m-m_1)}{\varkappa}$ *are the scalar curvatures of absolutely perpendicular non-singular* $m_1$*- and* $(m-m_1)$*-directions which are situated in the given m-direction, is independent of the particular choice of the* $m_1$*-direction, we have*

$$m_1 \underset{(m_1)}{\varkappa} + (m-m_1) \underset{(m-m_1)}{\varkappa} = m \underset{(m)}{\varkappa}. \quad \ldots\ldots \quad (2.70)$$

It should be noted that the last condition is equivalent to the condition

$$'U_{\gamma\lambda\mu\nu} = -\frac{4}{m-2}\, 'U_{[\varkappa[\mu}\, 'a_{\lambda]\nu]} \quad \ldots\ldots \quad (2.71)$$

Assume now that $V_n$ is a $C_n$, i.e. the affinor $U_{\varkappa\lambda\mu\nu}$ satisfies the relations (*2.6*), and denote by $\overline{K}_{\varkappa\lambda\mu\nu}$, $\overline{U}_{\varkappa\lambda\mu\nu}$ and $\overline{U}_{\lambda\mu}$ the $V_m$ components of $K_{\varkappa\lambda\mu\nu}$, $U_{\varkappa\lambda\mu\nu}$ and $U_{\lambda\mu}$, $V_m$ being defined in the preceding corollary. Taking the $V_m$-components of the both members of the equations (*2.1*) and (*2.6*) we obtain

$$\overline{U}_{\varkappa\lambda\mu\nu} = \overline{K}_{\varkappa\lambda\mu\nu} + \varkappa\, '\overset{2}{a}_{\varkappa\lambda\mu\nu} \quad \ldots\ldots \quad (2.72)$$

and

$$\overline{U}_{\varkappa\lambda\mu\nu} = -\frac{4}{n-2}\, \overline{U}_{[\varkappa[\mu}\, 'a_{\lambda]\nu]}. \quad \ldots\ldots \quad (2.73)$$

According to a well known theorem [13]) we have from (*2.72*)

$$\overline{U}_{\varkappa\lambda\mu\nu} = 'K_{\varkappa\lambda\mu\nu} + \varkappa\, '\overset{2}{a}_{\varkappa\lambda\mu\nu} = 'U_{\varkappa\lambda\mu\nu} + (\varkappa - \underset{(m)}{\varkappa})\, '\overset{2}{a}_{\varkappa\lambda\mu\nu} \quad \ldots \quad (2.74)$$

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[13]) See Ref. 10, V. II, p. 133, (14. 1).

from which we get by (2.73)

$$'U_{\varkappa\lambda\mu\nu} + (\varkappa - \underset{(m)}{\varkappa})\, '\overset{2}{a}_{\varkappa\lambda\mu\nu} = -\frac{4}{n-2}\, \overline{U}_{[\varkappa[\mu}\, 'a_{\lambda]\nu]} \quad . \quad . \quad . \qquad (2.75)$$

Multiplying (2.75) by $'a^{\varkappa\nu}$ and summing for $\varkappa$ and $\nu$ we obtain

$$'U_{\lambda\mu} + (\varkappa - \underset{(m)}{\varkappa})(1-m)\, 'a_{\lambda\mu} = \frac{m-2}{n-2}\, \overline{U}_{\lambda\mu} + \frac{1}{n-2}\, \overline{U}\, 'a_{\lambda\mu}, \quad . \qquad (2.76)$$

where

$$\overline{U} = \overline{U}_{\varkappa\nu}\, 'a^{\varkappa\nu}.$$

From (2.76) we get further

$$(\varkappa - \underset{(m)}{\varkappa})(1-m)\, m = \frac{m-2}{n-2}\, \overline{U} + \frac{m}{n-2}\, \overline{U} \,. \quad . \quad . \quad . \qquad (2.77)$$

from which

$$\overline{U} = -(\varkappa - \underset{(m)}{\varkappa})\, \frac{m\,(n-2)}{2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.78)$$

From (2.76) and (2.78) we obtain

$$\overline{U}_{\lambda\mu} = \frac{n-2}{m-2}\, 'U_{\lambda\mu} - \tfrac{1}{2}(\varkappa - \underset{(m)}{\varkappa})(n-2)\, 'a_{\lambda\mu}. \quad . \quad . \quad . \qquad (2.79)$$

and from (2.75)

$$'U_{\varkappa\lambda\mu\nu} = -\frac{4}{m-2}\, 'U_{[\varkappa[\mu}\, 'a_{\lambda]\nu]} \quad . \quad . \quad . \quad . \quad . \qquad (2.80)$$

Thus we get the lemma II: *The equation (2.80) is satisfied for every m-direction $2 \leq m \leq n$ at each point of the $V_n$ conformal to a flat space.*

This implies that every total geodesic subspace immersed in a $C_n$ must be a $C_m$ itself.

From the last lemma it follows also that *in a $C_n$ the relation (2.70) is satisfied for all integers m and $m_1$ satisfying* $4 \leq m \leq n$ and $2 \leq m_1 \leq m-2$.

Vice versa we shall prove. *If for a number m, $4 \leq m \leq n$, at each point of a $V_n$ the quantity*

$$m_1 \underset{(m_1)}{\varkappa} + (m - m_1) \underset{(m-m_1)}{\varkappa}$$

*is for every m-direction independent of the special choice of the non-singular $m_1$-direction inside the m-direction then the $V_n$ is a $C_n$.*

Take namely an orthogonal ennuple $\underset{1}{i}^{\lambda}, \underset{2}{i}^{\lambda}, \ldots \underset{n}{i}^{\lambda}$ and denote by $(h_1, h_2, \ldots h_n)$ an arbitrary permutation of the set $(1, 2, \ldots \mathrm{n})$. Consider further the m-direction defined by the vectors $\underset{h_1}{i}^{\lambda} \ldots \underset{h_m}{i}^{\lambda}$, then the orthogonal components of the RIEMANN curvature affinors of the $V_n$ and the $V_m$, defined in the corollary, satisfy the relations

$$K_{hijk} = {}'K_{hijk} \quad , \quad h, i, j, k = 1, \ldots, \mathrm{n} \quad . \quad . \quad . \qquad (2.81)$$

From our assumption follows (*2.71*), from which we can easily see that all orthogonal components $'K_{hijk}$ with four unequal indices vanish. Thus by (*2.81*) we have

$$K_{hijk} = 0 \qquad \text{for } h, i, j, k \quad ; \quad h, i, j, k = 1, \ldots, n$$

hence all orthogonal components of the RIEMANN curvature affinor of $V_n$ with four unequal indices vanish and by a well known theorem 14) the $V_n$ is conformal to a flat space 15).

We can now easily prove the following theorem of HAANTJES 16).

*A necessary and sufficient condition that a* $V_n$ *be a* $C_n$ *is that at every point the scalar*

$$m_1 \underset{(m_1)}{\varkappa} + m_2 \underset{(m_2)}{\varkappa} + \ldots + m_p \underset{(m_p)}{\varkappa} . \quad . \quad . \quad . \quad . \quad . \quad (2.82)$$

*should be independent of the choice of the set of p mutually absolutely perpendicular non-singular* $m_i$*-directions, where the integers*

$$m_i, \ (1 = 1, 2, \ldots p),$$

*are arbitrary exept for the conditions*

$$2 \leqslant m_i \leqslant n-2 \text{ and } \sum_{i|1}^{p} m_i = n$$

*and* $\underset{(m_i)}{\varkappa}$ *are scalar curvatures of corresponding* $m_i$*-directions.*

The scalar (*2.82*) is then equal to $n\varkappa$, $\varkappa$ being the scalar curvature of the space.

In fact, in the $C_n$ from the last but one theorem follows

$$\left.\begin{aligned} m_1 \underset{(m_1)}{\varkappa} + m_2 \underset{(m_2)}{\varkappa} + m_3 \underset{(m_3)}{\varkappa} + \ldots + m_p \underset{(m_p)}{\varkappa} &= (m_1 + m_2) \underset{(m_1+m_2)}{\varkappa} + m_3 \underset{(m_3)}{\varkappa} + \ldots m_p \underset{(m_p)}{\varkappa} = \\ &= \ldots\ldots\ldots\ldots = \\ &= n\varkappa \end{aligned}\right\} (2.83)$$

$\underset{(m_1+m_2)}{\varkappa}$ being scalar curvature of the corresponding $(m_1 + m_2)$-direction.

Inversely if

$$m_1 \underset{(m_1)}{\varkappa} + m_2 \underset{(m_2)}{\varkappa} + m_3 \underset{(m_3)}{\varkappa} + \ldots + m_p \underset{(m_p)}{\varkappa} = \tau, \quad . \quad . \quad . \quad (2.84)$$

where $\tau$ is independent of the choice of the assumed $m_i$-directions and if we do not change the $m_3$-, $m_4$-, $m_p$-directions, we get

$$m_1 \underset{(m_1)}{\varkappa} + m_2 \underset{(m_2)}{\varkappa} = \sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.85)$$

where $\sigma = \tau - m_3 \underset{(m_3)}{\varkappa} - \ldots - m_p \underset{(m_p)}{\varkappa}$ is independent of the choice of the $m_1$- and $m_2$-directions within the $(m_1 + m_2)$-direction. Thus from the last theorem we obtain that the $V_n$ must be a $C_n$.

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14) Ref. 10, V. II, p. 204.

15) This idea of the proof of this theorem is due to Prof. J. HAANTJES and different from my orginal one.

16) Ref. 5.

## Section 3. Principal m-vectors.

In order to shorten our work we shall confine ourselves in this section to the RIEMANNian spaces of an even number of dimensions, $2m$, and we shall consider $m$-vectors in them. There is little difficulty in generalizing the most of the following results to spaces of an odd number of dimensions.

Consider the projective $P_N$ associated with a point in $V_{2m}$ and coordinates in it defined by

$$\underset{\alpha}{v}^{\lambda_1\ldots\lambda_m}, \qquad \left.\begin{array}{l} \alpha = 1, 2, \ldots N+1, \\ \lambda_i = 1, 2, \ldots 2m. \end{array}\right\} \quad \ldots \ldots \quad (3.1)$$

In $P_N$ we have three $(N-1)$-dimensional quadrics defined by the tensors

$$a_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 \overset{m}{a}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \underset{\alpha}{v}^{\lambda_1\ldots\lambda_m} \underset{\beta}{v}^{\mu_1\ldots\mu_m}, \quad \ldots \quad (3.2)$$

$$U_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 \overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \underset{\alpha}{v}^{\lambda_1\ldots\lambda_m} \underset{\beta}{v}^{\mu_1\ldots\mu_m}, \quad \ldots \quad (3.3)$$

$${}^\circ U_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 {}^\circ\overset{m}{U}_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \underset{\alpha}{v}^{\lambda_1\ldots\lambda_m} \underset{\beta}{v}^{\mu_1\ldots\mu_m}. \quad \ldots \quad (3.4)$$

and we also have the affinor

$$I_{\alpha\beta} = \left(\frac{1}{m!}\right)^2 I_{\lambda_1\ldots\lambda_m\mu_1\ldots\mu_m} \underset{\alpha}{v}^{\lambda_1\ldots\lambda_m} \underset{\beta}{v}^{\mu_1\ldots\mu_m} \quad \ldots \ldots \quad (3.5)$$

which defines a quadric or null polarity according to $m$ being even or odd. We can easily prove that

$$a_{\alpha\beta} I^{\alpha\gamma} I^{\beta\delta} = a^{\gamma\delta} \quad \ldots \ldots \ldots \quad (3.6)$$

$$U_{\alpha\beta} I^{\alpha\gamma} I^{\beta\delta} = \varrho^2\, {}^\circ U^{\gamma\delta} \quad \ldots \ldots \ldots \quad (3.7)$$

and

$$f^\alpha = \varrho^3\, {}^\circ f_\delta I^{\delta\alpha} \quad \ldots \ldots \ldots \quad (3.8)$$

where

$${}^\circ f_\delta = \frac{1}{m!}\, {}^\circ f_{\lambda_1\ldots\lambda_m} \underset{\delta}{v}^{\lambda_1\ldots\lambda_m}. \quad \ldots \ldots \ldots \quad (3.9)$$

The deviation of an $m$-vector is now given by

$$\omega = -\frac{U_{\alpha\beta} f^\alpha f^\beta}{a_{\alpha\beta} f^\alpha f^\beta} \quad \ldots \ldots \ldots \quad (3.10)$$

At a point the finite maxima and minima of $\omega$ defined by (3.10) are given by the $m$-vectors satisfying

$$(U_{\alpha\beta} + \omega\, a_{\alpha\beta}) f^\alpha = 0 \quad \ldots \ldots \ldots \quad (3.11)$$

at that point.

The $m$-vectors which satisfy equations (*3.11*) for some value of $\omega$ will be called the *principal m-vectors.*

Multiplying (3.11) by $\varrho I^{\beta\gamma}$ and summing for $\beta$ we have from *(3.8)*, *(3.6)* and *(3.7)*

$$({}^{\circ}U^{\alpha\beta} + \varrho^2\, \omega\, a^{\alpha\beta})\, {}^{\circ}f_{\alpha} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (3.12)$$

or equivalently

$$({}^{\circ}U_{\alpha\beta} + \varrho^2\, \omega\, a_{\alpha\beta})\, {}^{\circ}f^{\alpha} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (3.13)$$

Thus we have: *If an m-vector* $f^{\alpha}$ *satisfies equation (3.11) for a characteristic root* $\omega$, *then its dual* ${}^{\circ}f^{\alpha}$ *satisfies (3.13) for the characteristic root* $\varrho^2\omega$.

Consider now an EINSTEINian $V_{2m}$. From (2.59), (3.3) and (3.4) we have in this case

$${}^{\circ}U_{\alpha\beta} = \varrho^2\, U_{\alpha\beta}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.14)$$

and (3.13) becomes

$$(U_{\alpha\beta} + \omega\, a_{\alpha\beta})\, {}^{\circ}f^{\alpha} = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.15)$$

Hence: *In an* EINSTEIN*ian* $V_{2m}$ *the dual of a principal m-vector is also a principal m-vector, the characteristic roots are the same for both m-vectors.*

The *m*-vectors

$$\bar{f}^{\alpha} = \tfrac{1}{2}\,(f^{\alpha} + {}^{\circ}f^{\alpha}) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.16)$$

and

$$\bar{\bar{f}}^{\alpha} = \tfrac{1}{2}\,(f^{\alpha} - {}^{\circ}f^{\alpha}) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.17)$$

satisfy the relations

$${}^{\circ}\bar{f}^{\alpha} = \bar{f}^{\alpha} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.18)$$

$${}^{\circ}\bar{\bar{f}}^{\alpha} = -\bar{\bar{f}}^{\alpha} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.19)$$

and are therefore self-dual. If $f^{\alpha}$ is principal *m*-vector so are $\bar{f}^{\alpha}$ and $\bar{\bar{f}}^{\alpha}$. Thus: *If at a point of an* EINSTEIN*ian* $V_{2m}$ *the characteristic roots of (3.11) are distinct then there are* $(N+1)$ *linearly independent principal m-vectors at the point, they are all self-dual, half of them satisfying (3.18) and the other half (3.19).*

Consider now a $C_{2m}$. From *(2.60)* we have

$${}^{\circ}U_{\alpha\beta} = -\varrho^2\, U_{\alpha\beta} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.20)$$

and equation *(3.13)* becomes

$$(U_{\alpha\beta} - \omega\, a_{\alpha\beta})\, {}^{\circ}f^{\alpha} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.21)$$

Hence: *In a* $C_{2m}$ *the dual of a principal m-vector is also a principal m-vector, and corresponding characteristic roots differ only in sign.*

Thus in a $C_{2m}$ we can arrange the characteristic roots of *(3.11)* in pairs in such a way that the roots in each pair differ only in sign.

By (2.60) and (2.46), we have in $C_{2m}$

$$\overset{m}{U}_{\lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_m} = -\frac{m!}{m-1}\, U_{[\lambda_1[\mu_1}\, a_{\lambda_2\mu_2} \ldots a_{\lambda_m]\mu_m]} \quad . \; . \quad (3.22)$$

Consider now the RICCI principal directions in $C_{2m}$, the elementary divisors of the RICCI characteristic equation being assumed simple. Let $\underset{i}{k}$

be their mean curvatures. Take the orthogonal ennuple of RICCI principal directions at a point of $C_{2m}$ to be the basis of the coordinate-system in the $P_N$ associated with this point. It is easily seen from (*3.22*) that the coordinate-$m$-vectors of this system are principal $m$-vectors and that their coresponding characteristic roots $\underset{\alpha}{\omega}$, $(\alpha = 1, 2, \dots N+1)$, are

$$\underset{\alpha}{\omega} = \frac{1}{m(m-1)} \sum_{p}^{1\dots m} \underset{\alpha_p}{k} - \frac{2m-1}{m-1} \varkappa, \quad . \; . \; . \; . \; . \qquad (3.23)$$

where $\underset{\alpha_p}{k}$ are mean curvatures of the corresponding RICCI principal directions situated inside of the $m$-direction defined by the $\alpha$-th principal $m$-vector.

But from (*2.21*) we get

$$\underset{\alpha}{\omega} = \underset{\alpha}{\varkappa} - \varkappa, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (3.24)$$

where $\underset{\alpha}{\varkappa}$, $(\alpha = 1, 2, \dots N+1)$, is the scalar curvature of the $m$-direction defined by the $\alpha$-th principal $m$-vector.

Hence

$$\underset{\alpha}{\varkappa} = \frac{1}{m(m-1)} \sum_{p}^{1\dots m} \underset{\alpha_p}{k} - \frac{m}{m-1} \varkappa \quad . \; . \; . \; . \; . \; . \qquad (3.25)$$

Thus we have the theorem:

*If in a* $C_{2m}$ *the* RICCI *elementary divisors are assumed simple then the m-vectors defined by the m-directions composed of every set of m* RICCI *principal directions are the principal m-vectors and the scalar curvatures of these m-directions are given by (3.25).*

Consider now a $V_{2m}$ which is immersible in a flat $(2\,m + 1)$-dimensional space. In this case we have [17]

$$K_{\varkappa\lambda\mu\nu} = -2\varepsilon\, h_{[\varkappa[\mu}\, h_{\lambda]\nu]}, \quad . \; . \; . \; . \; . \; . \; . \qquad (3.26)$$

where $\varepsilon = \pm 1$ and $h_{\varkappa\mu}$ is the second fundamental tensor. We have further

$$\overset{m}{U}_{\lambda_1\dots\lambda_m\mu_1\dots\mu_m} = -m!\,\varepsilon\, h_{[\lambda_1[\mu_1}\, h_{\lambda_2\mu_2}\, a_{\lambda_3\mu_3}\dots a_{\lambda_m]\mu_m]} + \varkappa\, \overset{m}{a}_{\lambda_1\dots\lambda_m\mu_1\dots\mu_m}. \qquad (3.27)$$

The principal directions of the tensor $h_{\lambda\mu}$ are the same as the RICCI principal directions. Let $\underset{i}{h}$ be the principal normal curvatures of $V_{2m}$. Consider, in the $P_N$ associated with a point of the $V_{2m}$ under consideration, the basic coordinate-system having the orthogonal ennuple of principal directions of $h_{\lambda\mu}$ as its basis. It is easy to show from (*3.27*) that the coordinate $m$-vectors of this system are the principal $m$-vectors and that the characteristic roots of (*3.11*) are

$$\underset{\alpha}{\omega} = \frac{\varepsilon}{m(m-1)} \sum_{p \neq q}^{1\dots m} \underset{\alpha_p}{h}\, \underset{\alpha_q}{h} - \varkappa; \quad . \; . \; . \; . \; . \qquad (3.28)$$

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[17] Ref. 10, V. II, p. 76.

From (3.24) and (3.28) we have

$$\underset{\alpha}{\varkappa} = \frac{\varepsilon}{m(m-1)} \sum_{p \neq p}^{1\ldots m} \underset{\alpha p}{h}\, \underset{\alpha q}{h} \quad . \; . \; . \; . \; . \; . \; . \quad (3.29)$$

Thus: *In a* $V_{2m}$ *which is immersible in a* $(2m+1)$*-dimensional flat space, the m-vectors defined by all m-directions of the principal ennuple of the tensor* $h_{\lambda\mu}$ *are the principal m-vectors; the scalar curvatures*

$$\underset{\alpha}{\varkappa}, \ (a = 1 \ldots N + 1),$$

*of these m-directions, are given by (3.29), in which* $\underset{\alpha p}{h}$, $(p = 1, \ldots, m)$. *are corresponding principal normal curvatures.*

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*Inleiding.*

Sinds HARTZ en MILTHERS in 1901 voor de eerste maal in laatglaciale sedimenten op het Deense eiland Seeland een tijdelijke verbetering van het klimaat konden aantonen, die de Allerød-oscillatie werd genoemd, is deze voor de correlatie zo belangrijke klimaatsschommeling op vele plaatsen in Europa waargenomen. Zo werd het voorkomen van de Allerød oscillatie aangetoond onder meer bij Hamburg door SCHÜTRUMPF (o.a. 1936), in Oostpruisen door GROSZ (1937), bij Bremen door OVERBECK en SCHNEIDER (1938), in Ierland door JESSEN en FARRINGTON (1938), in het Untereichsfeld, ten Zuiden van de Harz, door STEINBERG (1944), in Frankrijk door G. en C. DUBOIS (1944), in het Bodenmeer gebied door INGE MÜLLER (1947) en in het Lake District in Engeland door WINIFRED PENNINGTON (1947). Voorts werd onze kennis van flora en klimaat in het Laatglaciaal belangrijk uitgebreid door een aantal publicaties van de hand van IVERSEN (o.a. 1946, 1947). In Nederland was het vooral FLORSCHÜTZ (o.a. 1939, 1941, 1944), die zich met de vegetatie-ontwikkeling in het Laatglaciaal heeft beziggehouden. Hij heeft enige diagrammen gepubliceerd die gedeeltelijk gebaseerd waren op de analysen van laatglaciaal veen. Het resultaat van zijn onderzoek stelde hem in staat, voor ons land een schema van de vegetatie-ontwikkeling te ontwerpen (zie de tabel). FLORSCHÜTZ (1939) was het ook, die voor de eerste maal de mogelijkheid opperde van het voorkomen van afzettingen uit de Allerødtijd in Nederland. Later wees WATERBOLK (1947) eveneens hierop. Tot nu toe heeft men echter de Allerød-oscillatie in Nederland nog niet met zekerheid aan kunnen tonen. De oorzaak daarvan moet gezocht worden in het feit, dat slechts een gering aantal laatglaciale meerafzettingen uit Nederland bekend is, in tegenstelling tot b.v. Denemarken, waar deze afzettingen zeer veelvuldig zijn aangetroffen. Meerafzettingen blijken n.l. zeer scherp veranderingen in het klimaat weer te geven, door een daarmede parallel gaande verandering in de sedimentatie: meer minerogeen tijdens slechtere, meer organogeen tijdens betere klimaatsomstandigheden. Bovendien zijn de locale invloeden op de polleninhoud van het sediment, zoals die vooral in veen optreden tengevolge van de vegetatie ter plaatse, aanmerkelijk geringer in meerafzettingen. Dit is van groot belang, daar de polleninhoud van laatglaciale venen beïnvloed kan zijn door de locale vegetatie van Cyperaceae. De pollenspectra geven dan geen zuiver algemeen beeld van de plantengroei ten tijde van de afzetting, maar representeren de locale begroeiïng, gesuperponeerd op de regionale. Deze verwikkeling zal eerst ten volle te ontwarren zijn, wanneer met behulp van de analyse van meerafzettingen beter inzicht zal zijn verkregen in de regionale vegetatie-ontwikkeling in Nederland. Door de onderzoekingen van enkele pollenanalytici, waaronder IVERSEN op de voorgrond treedt, is het door een vergelijkende studie van de morphologie van pollen en sporen mogelijk geworden, om naast de stuifmeelkorrels van bomen een groot aantal soorten kruidenpollen en sporen te determineren. Dit kruidenpollen blijkt nu juist in laatglaciale sedimenten van uitermate groot belang te zijn, niet alleen quantitatief maar ook qualitatief. In de eerste plaats zijn de zuivere botanische resultaten van het kruidenpollenonderzoek natuurlijk van veel gewicht voor de reconstructie der vegetatie. Daarnaast blijken die resultaten echter ook de stratigrafie te kunnen dienen, daar een bepaald soort kruidenpollen of een associatie van verschillende soorten alleen of hoofdzakelijk in bepaalde zones van het Laatglaciaal voor kan komen. IVERSEN heeft nu voor het weergeven van de uitkomsten der analysen van laatglaciale afzettingen een geheel nieuw diagram-type ontworpen, dat meer beantwoordt aan de bijzondere eisen, die aan een duidelijke afspiegeling van de vegetatie-veranderingen in die tijd mogen worden gesteld. Als basis van procentberekening neemt hij de som van al het pollen van bomen, anemophiele kruiden en Ericaceae. Alle stuifmeelkorrels van andere planten en alle sporen blijven buiten deze „pollensom". Voor het verkrijgen van één spectrum worden ongeveer 500 tot de „pollensom" gerekende stuifmeelkorrels geteld. Dit is noodzakelijk, om de gewenste statistische zekerheid voor de percentages der verschillende soorten van het kruidenpollen te bereiken. Het zal duidelijk zijn, dat het op deze wijze samenstellen van een diagram zeer veel tijd in beslag neemt. De resultaten blijken er echter ten volle tegen op te wegen.

### *Vindplaats en aard van het onderzochte materiaal.*

In April 1948 werd, met het doel een laatglaciale meerafzetting te vinden, geboord met de Dachnovsky-sonde aan de rand van de verlandingszone aan de Westoever van het Hijkermeer. Dit is gelegen bij Hijken ten Zuid-Westen van Hooghalen in de provincie Drente. Deze plaats werd uitgekozen naar aanwijzingen van Prof. FLORSCHÜTZ. Het meertje, dat afmetingen heeft van ongeveer 200 m × 150 m, is een van de vele kommen op het Drentse plateau, hier omringd door zandafzettingen, op de geologische kaart als fluvio-glaciaal aangegeven. In de naaste omgeving komt het keileem aan de oppervlakte voor. In het midden staat waarschijnlijk ongeveer 5 m water, reeds een opmerkelijke diepte voor een dergelijke kom. Bovendien werd nu op de boorplaats een sediment-pakket van 6,5 m aangetroffen, voor het onderliggende zand bereikt werd, waaronder dan nog de keileem te verwachten is. Op zijn minst moet de diepte in het midden oorspronkelijk dus 11 m bedragen hebben. Het is wel zeer waarschijnlijk, dat we hier met een kom te maken hebben, die zijn ontstaan aan een in de grondmoraine achtergebleven massa gletscherijs te danken heeft.

Aangetroffen werden (dieptecijfers globaal; zie verder het diagram)

| | | | |
|---|---|---|---|
| 0— | ± 100 | cm | veen van de verlandingszone; |
| ± 100— | 510 | „ | detritus-gyttja; |
| 510— | 580 | „ | klei-gyttja, met uiterst fijn „zand"; |
| 580— | 600 | „ | detritus-gyttja; |
| 600— | 630 | „ | klei-gyttja; |
| 630— | ? | „ | zand. |

Al dadelijk rees het vermoeden, dat het in de kleigyttja ingeschakelde laagje detritusgyttja zou corresponderen met de Allerød-oscillatie, daar deze afwisseling van meer minerogene en meer organogene afzettingen onder analoge omstandigheden elders karakteristiek is hiervoor. Het microbotanische onderzoek heeft dit vermoeden geheel bevestigd. De onder en boven de Allerødlaag aanwezige sedimenten bleken naar hun polleninhoud overeen te komen met resp. de Oudere en Jongere Dryas-lagen uit Denemarken. Hoewel overblijfselen van Dryas octopetala of dergelijke toendraplanten in deze lagen in Nederland nog niet aangetoond zijn, en het zelfs min of meer twijfelachtig is, of deze plant ten tijde van de afzetting der bovenste kleigyttja in Nederland voorkwam, menen we toch deze namen, als zuiver stratigrafische, over te moeten nemen. Bovendien hebben zij in de Europese literatuur reeds algemeen ingang gevonden.

### *Wijze van werken.*

Bij de bereiding van het materiaal voor de analysen werd gebruik gemaakt van de verkorte methode ERDTMAN, zoals die bij de Deense Geologische Dienst toegepast wordt, steeds voorafgegaan door een behandeling met KOH en HF. Verder werden de berekeningen der procenten geheel uitgevoerd op de reeds in de inleiding vermelde methode van IVERSEN. Zodoende werd een grote mate van vergelijkbaarheid met de Deense diagrammen verkregen.

### *Secundair pollen.*

Bij de analyse van het materiaal werden in een aantal monsters, afkomstig uit het laatglaciale deel van de afzettingen, enkele stuifmeelkorrels van warmteminnende bomen aangetroffen, die in dit milieu allerminst verwacht konden worden. IVERSEN (1936) heeft het zelfde verschijnsel in Denemarken waargenomen, alleen in veel sterkere mate. Hij toonde aan, dat deze stuifmeelkorrels — waaronder zich typisch tertiaire en interglaciale typen bevinden — tezamen met het minerogene materiaal uit het keileem afkomstig moeten zijn. Steeds werden ze ook begeleid door een waarschijnlijk uit het Tertiair afkomstig marien micro-organisme, waaraan hij de naam Hystrix gaf, en dat eveneens in het keileem voor bleek te komen. Door een vernuftige correctie-methode wist hij, met behulp van het zeker secundaire pollen, ook het ingespoelde pollen dat niet direct als zodanig herkenbaar was (b.v. van Pinus en Betula), te elimineren. Zo bleken bijvoorbeeld bijna alle in de Oudere Dryas-afzettingen voorkomende Pinus korrels secundair te zijn. Deze methode kon helaas op ons materiaal niet worden toegepast, maar gelukkig bleek de hoeveelheid zeker secundair pollen zeer gering te zijn, zodat de verontreiniging slechts een zeer geringe invloed op het diagram gehad kan hebben.

Wij hebben gemeend het weinige pollen van warmteminnende houtige gewassen, dat in ons laatglaciale materiaal gevonden werd, niet in het diagram te moeten aangeven, maar in een hieronder volgende tabel te moeten vermelden, daar wij het voor een groot deel als secundair beschouwen, en wel om de navolgende redenen:

1. Het voorkomen ervan in de min of meer minerogene afzettingen, terwijl het in de meer organogene afzettingen bijna geheel ontbreekt. Dit betekent dus een afname bij klimaatsverbetering.

(Bij een verbetering van het klimaat treedt namelijk een meer volledige begroeiïng op, waardoor de erosie minder sterk, en dus de lacustrine sedimentatie minder minerogeen wordt.)

2. Het eveneens voorkomen van stellig secundair pollen, bijvoorbeeld van Tsuga, Rhus, Platycarya, Carya, Abies en Picea.

3. Het samengaan met „Hystrix".

4. De conservatietoestand, die vaak vrij slecht is, in tegenstelling tot het overige aanwezige pollen (uitgezonderd sommige Pinus korrels, die waarschijnlijk eveneens secundair zijn), terwijl bovendien de kleur afwijkt van het overige materiaal, wat doet denken aan een voorafgaande insluiting in veen of bruinkool.

| Monster | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Corylus | 1 | 3 | 3 | 3 | 1 | 1 | — | — | — | 2 | 1 | 2 | 5 | 2 | 3 | (1) | Aantal pollenkorrels |
| Quercus | — | 1 | — | 1 | 1 | — | — | — | — | — | — | — | 1 | — | — | — | |
| Alnus | 1 | — | — | 2 | — | — | — | — | — | — | — | — | — | — | — | — | |
| Picea/Abies | 1 | 1 | — | — | — | — | — | — | — | — | — | 1 | — | 1 | — | — | |
| Tertiaire typen | — | 2 | 1 | 1 | — | — | — | — | — | — | — | — | 1 | — | — | — | |
| Hystrix | — | × | × | ? | — | — | — | — | — | — | — | × | — | × | — | — | |

Helaas is de Allerød-laag niet geheel zuiver organogeen, want in het bovenste deel van de detritusgyttja komen kleine mineraal-partikeltjes voor. Daardoor is het niet uit te maken, of een enkele Corylus-korrel daarin primair voorkomt, een mogelijkheid, die misschien niet geheel uitgesloten mag worden.

We zullen na de analyse van niet geheel zuiver organogene en van minerogene sedimenten van laatglaciale ouderdom, de uiterste voorzichtigheid moeten betrachten bij het trekken van conclusies uit het voorkomen van een enkele korrel van meer-warmteminnende bomen. Voorts is er de mogelijkheid van transport over grote afstand, dat in boomarme gebieden op kan treden, van het beboste achterland uit. Mogelijk zijn ook daardoor enkele Corylus-korrels in de sedimenten van het Hijkermeer terecht gekomen.

Over het voorkomen van Corylus-pollen in laatglaciale afzettingen zie men ook het laatste onderdeel van dit artikel.

### *Algemeen overzicht van de vegetatie-ontwikkeling.*

Het diagram is verdeeld in drie afdelingen. A is het algemene diagram, tonend van links uitgezet de curven van de bomen en struiken, en van rechts uitgezet de percentages van de som der anemophiele kruiden en van de Ericaceae. Deze beide groepen vormden tezamen de basis voor de procentberekening. Corylus is, zoals gebruikelijk, buiten de som gehouden. B geeft de curven van de anemophiele kruiden en van Empetrum afzonderlijk, en C geeft hetzelfde voor de niet in de „pollensom" begrepen planten, te beginnen met Juniperus. Deze laatste is, om resultaten te verkrijgen die vergelijkbaar zouden zijn met de gepubliceerde Deense diagrammen, buiten de „pollensom" gehouden. Geheel rechts zijn de waarden opgegeven van de boompollenfrequentie (Tree Pollen Frequency, T.P.F.), het gemiddeld aantal boompollen-korrels per preparaat.

In A komt duidelijk de afwisseling van een open landschap met meer of minder dichte bossen tot uitdrukking. In het onderste deel toont dit diagram ons het beeld van een open landschap met slechts weinig bomen. Bovendien is waarschijnlijk hier een groot deel van het Betula-pollen afkomstig van Betula nana, naar een oppervlakkige grootte-beoordeling. Om hiervan meer zekerheid te krijgen zal een statistische meting van de korrels noodzakelijk zijn. De boompollenfrequentie is in de onderste monsters gering. We zouden het uit dit deel van het diagram sprekende vegetatie type, met enig voorbehoud, een „niet geheel boomloze toendra" willen noemen. Wij hebben deze uitdrukking gekozen, in de veronderstelling, dat hier, evenals in Denemarken, toentertijd een landschap aanwezig was, dat alhoewel niet geheel boomloos, de meeste overeenkomst met een toendra zal hebben gehad. Met nadruk vestigen wij echter de aandacht op de omstandigheid, dat van echte toendraplanten tot dusver geen overblijfselen gevonden zijn.

Hierna vertoont het diagram een duidelijke stijging van de Betula-percentages, samengaand met een vermindering van de kruiden-procenten. We noemen het in die tijd aanwezige vegetatietype, in navolging van IVERSEN, een „parktoendra", die we ons moeten voorstellen als een open landschap met verspreide boomgroepen.

Direct daarop volgt een nieuw, en ditmaal zeer hoog maximum van het kruidenpollen. De weinige Betula-pollenkorrels zijn ook hier bijna alle opvallend klein, terwijl de T.P.F. zeer gering is. Het lijkt aannemelijk, hier tot het bestaan hebben van een toendra te besluiten, daarbij dezelfde gedachtengang volgend, die tot de keuze van de uitdrukking „niet geheel boomloze toendra" heeft geleid, en met inachtneming van hetzelfde voorbehoud.

Hierna treedt een plotselinge en sterke teruggang op van de kruidenlijn, gepaard gaande met een scherpe stijging van de Betula-curve, en met een veel grotere T.P.F. Iets hoger in het diagram neemt het Pinuspercentage plotseling toe en vervolgens geleidelijk weer af, terwijl het kruidenprocent opnieuw groter wordt. Blijkbaar was het terrein om het meer in de tijd van het lage kruidenpercentage bebost, in tegenstelling met daarvoor en daarna. Ook de sedimentatie wijst op een klimaatsverbetering, want in deze bostijd was zij hoofdzakelijk organogeen, na tevoren minerogeen geweest te zijn. De bossen worden, gezien het hoge kruidenpercentage, weer lichter, maar ditmaal komt het niet verder dan tot de vorming van een „parktoendra". De Empetrum-heide moet in deze tijd een grote uitbreiding gehad hebben.

Ten slotte begint nu, zoals uit het bovenste deel van het diagram blijkt, de definitieve bebossing. Spoedig nemen Pinus en Corylus een belangrijke plaats in de bossen in, terwijl ook de componenten van het Quercetum mixtum verschijnen.

### *Parallelisatie en zone-indeling.*

Vergelijkt men nu het diagram van het Hijkermeer met de Deense diagrammen van de hand van IVERSEN, dan blijkt met een oogopslag, dat in het onze een duidelijke Allerødzone aanwezig is. Ook overigens is een goede parallelisatie met de Deense diagrammen mogelijk, niet alleen wat het algemene verloop aangaat, maar ook wat betreft het voorkomen van bepaalde soorten kruidenpollen. We hebben dan ook niet geaarzeld, voor ons diagram de zone-indeling van IVERSEN te gebruiken (zie hiervoor het diagram en de overzichtstabel).

Het lijkt nuttig, om hier, waar voor de eerste maal de Allerød-oscillatie met zekerheid in Nederland herkend is, de waarnemingen op te sommen, die tot haar aanwezigheid deden besluiten. Deze zijn:

1. Een wijziging in de sedimentatie: een organogene afzetting, een laag detritus-gyttja, ingeschakeld in een pakket hoofdzakelijk minerogene sedimenten van laatglaciale ouderdom, wijzend op een tijdelijke klimaatsverbetering.

2. Het plotseling sterk afnemen van de kruidenpollenpercentages in de detritus-gyttja-laag en het toenemen van de boompollenprocenten, met stijging van de boompollenfrequentie en een daling van de Salix-percentages. Dit alles eveneens wijzend op een tijdelijke klimaatsverbetering.

3. De overeenstemming van de onder en boven deze laag liggende sedimenten — wat kruidenpolleninhoud betreft — met de sedimenten onder en boven de Allerød-laag in Denemarken, resp. overeenkomende met de Oudere Dryas-laag en de Jongere Dryas-laag. Zo komen b.v. voor Hippophaë en Helianthemum in de er onder liggende, Empetrum in de er boven liggende laag.

T. VAN DER HAMMEN: *De Allerød-oscillatie in Nederland.*

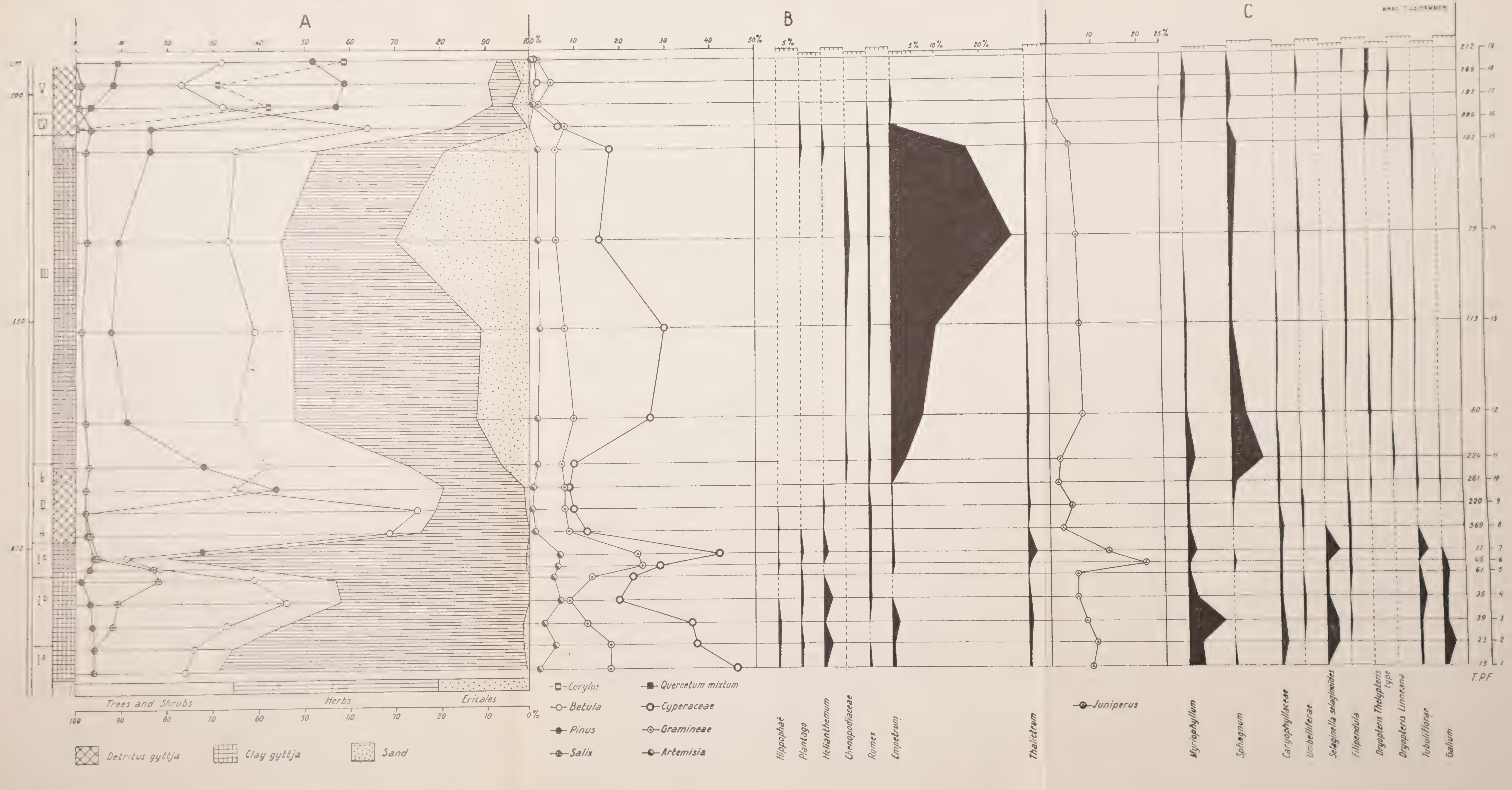


Diagram from the Hijkermeer (lake of Hijken), in the province of Drente, Holland. The som of calculation of the pollen percentages consists of the pollen of all anemophile plants (with the exception of the water-plants).

The diagram consists of three parts:

A. The general part.

B. Pollen curves for the anemophile herbs (together with Hippophaë and Empetrum), which are included in the som of calculation (striped and spotted part of A).

C. Curves for the pollen and spores of plants which are not included in the som.

(T.P.F. = Tree Pollen Frequency number of treepollengrains per slide 18 × 18 mm).

Hiermede zijn dus tevens de redenen van parallelisatie van de onder en boven onze Allerød-laag aanwezige laatglaciale sedimenten met IVERSENS zones I en III aangegeven.

Er blijkt nu in onze zone I nog een kleine oscillatie voor te komen, die wel is waar van veel geringere betekenis is dan de Allerød, doch niettemin een opvallend karakter heeft. IVERSEN vond een dergelijke klimaatsschommeling op geheel analoge plaats in het diagram in afzettingen uit het Bøllingmeer in Midden-Jutland (IVERSEN, 1947), die hij Bølling-oscillatie noemde. Zowel bij onze oscillatie als bij die van Bølling is de klimaatsterugslag aan het eind veel scherper dan de klimaatsverbetering aan het begin, die geleidelijker geschiedde, bij beide uitkomend in het verloop van de Betula- en de kruidenlijn. Het laat dan ook geen twijfel, dat beide oscillaties identiek zijn. Daardoor zijn we in staat, hier ook IVERSENS onderverdeling van de Oudere Dryastijd te gebruiken, n.l.

I*c* Vroegere Dryas-tijd
I*b* Bølling-oscillatie
I*a* Vroegste Dryas-tijd.

We gebruiken deze benamingen, hoewel zij minder fraai zijn, inplaats van Oudere en Oudste Dryas-tijd, om verwarring met de naam Oudere Dryas-tijd voor de gehele zone I te voorkomen. De Allerød-tijd kan bij ons verdeeld worden in een oudere fase met zuivere Betula-bossen, en een jongere fase met Pinus-Betula-bossen. Merkwaardig is de plotseling steil opkomende Pinus-lijn. Een geheel analoog verloop heeft deze lijn op Bornholm (IVERSEN, schr. med.). Een indeling in tweeën van de Allerødtijd op grond van een kleine klimaatsschommeling, zoals die in de Deense diagrammen vaak uitkomt, is in ons diagram niet aan te tonen. Volgens IVERSEN gaat het hier om een kleine oscillatie, die alleen in een zeker randgebied om de gletschers uit zal komen.

Op zone III volgt, met de definitieve klimaatsverbetering en bebossing, zone IV, het Praeboreaal, waarvan het begin samenvalt met een verandering in sedimentatie.

Slechts één monster (16) valt in deze zone, die hier dan ook slecht ontwikkeld is. In dit monster werd één Corylus korrel aangetroffen. De zone van de zuivere Pinus-Betula-bossen schijnt dus in ons diagram te ontbreken; het is evenwel onzeker, welke waarde aan de vondst van slechts één Corylus-korrel gehecht moet worden.

De monsters 17, 18 en 19 behoren tot het onderste deel van het Boreaal, zone V, met vrij hoge Corylus-percentages en opkomst van de componenten van het Quercetum mixtum.

# **Botany.** — *De $F_4$-zaadgeneratie van 1936 na kruisingen van twee zuivere lijnen van Phaseolus vulgaris.* I. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of October 30, 1948.)

Het uitgangsmateriaal voor deze onderzoekingen, die in 1932 werden begonnen, bestond uit enige bonen van twee zuivere lijnen. (Genetica, 1934). De bonen van de I-lijn zijn lang, breed en dun, die van de II-lijn zijn kort, iets minder breed en dik (Deze Proceed. 1948, Vol. 51, No. 2 en 3, Pl. 1). De bonen van de eerste bastaardgeneratie, $F_1$, vertonen matroclinie, te verklaren uit de moederlijke zaadhuid van de $F_1$-bonen (Genetica, 1947). In de $F_2$-zaadgeneratie treedt splitsing op, die getemperd wordt door de uniforme zaadhuid van de $F_1$-planten (Genetica, 1947, 2). De $F_3$-zaadgeneratie wordt gevormd uit de bonenopbrengsten van $F_2$-planten en deze kunnen zeer verschillen (Genetica, 1947, 2). Bij de kruisingen vertonen de bonen enige dominantie van de grote afmetingen over de kleine. Voor de verklaring van onze resultaten nemen we de werking van polymere factoren aan.

Deze mededeling over de $F_4$-zaadgeneratie van 1936, die terug te voeren is tot kruisingen van 1933 sluit aan bij een publicatie in Genetica (1948) over de $F_4$-zaadgeneratie van 1935, die tot kruisingen van 1932 teruggevoerd kan worden.

We groepeerden het materiaal volgens de formule van de uitgangsbonen van $F_3$-1935 voor de $F_4$-bonenopbrengsten van 1936. Volgens het vereenvoudigde tetrahybride schema onderscheiden we 8 classen (1947; deze Proceedings, Vol. 50, No. 7).

*Cl 1a.* De formule van de uitgangsbonen is $L_1$ $L_2$ B Th, d.w.z., alle 3 afmetingen zijn zeer groot. 14 gevallen.

In één geval is de formule van de gemiddelden van de bonenopbrengst (pl. 331, tab. 1 en 1a) ook $L_1$ $L_2$ B Th.

Van de bonenopbrengst van de pl. 84 (tab. 21, 1947, Genetica) d.i. de plant, waarvan de uitgangsboon voor pl. 331 genomen is, is de formule van de gemiddelden L B Th. De uitgangsboon voor pl. 84 is van pl. 82, $F_2$-1934, en heeft de form. L b Th, cl 3; de breedte blijft iets beneden de grenswaarde (b = 8.4 mm). De gemiddelde dikte van de bonenopbrengst van pl. 331 is groter dan we bij bonenopbrengsten van de I-lijn van 1936 aantreffen; het gemiddelde gewicht is veel groter (tab. 4a). Van de individuele bonen van de bonenopbrengst van pl. 331 is de grootste lengte l = 17.5 mm, de grootste dikte d = 7.3 mm. Van de uitgangsboon van pl. 84 voor pl. 331 beantwoordt het genotype in hoge mate aan de form. $L_1$ $L_2$ B Th.

De formule van de gemiddelden van de bonenopbrengsten van de overige 13 gevallen is in 2 gevallen (pl. 274 en 404) $L_1$ $l_2$ B Th, cl 1b, in 6 gevallen (pl. 275, 277, 300, 306, 309 en 336) $L_1$ $l_2$ B th, cl 2b, in 2 gevallen (pl. 267 en 278, blz. II 76) L B th, cl 4, in één geval (pl. 293) L b Th, cl 3 (grensgeval van cl 1) en in 2 gevallen l b th, cl 8. Uit deze opsomming en uit het verdere onderzoek en de classificatie, blijkt, dat we met verwante gevallen te doen hebben. In het centrum staan de 6 gevallen met de form. L B th, cl 2, der gemiddelden. Van één van deze gevallen geven we een korte beschrijving (pl. 309, tab. 1 en 1a).

De uitgangsboon is 2 p l b van pl. 73, $F_3$-1935 (tab. 1). Van de gemiddelden van pl. 73 is de formule $L_1$ $L_2$ B th als van de I-lijn; de gemiddelde lengte is zeer groot (l = 16.0 mm, tab. 21, 1947, Genetica). De uitgangsboon voor pl. 73, $F_3$-1935, is 4 p l b van pl. 81, $F_2$-1934; de formule is $L_1$ $l_2$ B Th; de lengte is zeer groot, heeft de grenswaarde (l = 15.5 mm, tab. 21, 1947). De formule van de gemiddelden is eveneens L B Th.

De bonenopbrengst van 35 bonen van pl. 73 heeft een zeer grote gemiddelde lengte en breedte ($l_m$ = 15.9, $b_m$ = 9.5 mm) en een kleine gemiddelde dikte ($th_m$ = 6.3 mm); de formule is $L_1$ $L_2$ B th, als van de I-lijn. Het gemiddelde gewicht is zeer groot ($w_m$ = 66 cg). Vergeleken met de gemiddelden van bonenopbrengsten van de I-lijn, zijn de gemiddelde indices (60, 40 en 66 iets te hoog; ook het gemiddelde gewicht is iets te groot (tab. 2a). Onder de individuele bonen zijn er twee, zoals we ze onder de bonen van de I-lijn niet aantreffen. Met de afmetingen van de uitgangsboon van pl. 73 voor pl. 309 komen die van enkele bonen van de I-lijn van 1935 als hoge uitzondering overeen (tab. 3a). Het gewicht is echter groter (w = 101 cg); de 2 overige bonen van de peul hebben ook zeer grote afmetingen en een zeer groot gewicht (w = 97 en 92 cg).

De gemiddelden van de bonenopbrengst van 38 bonen van pl. 309 zijn groot (tab. 1), de formule is L B th als van de I-lijn. De gemiddelde breedte is zeer groot; daardoor is de gemiddelde L B-index hoog en de gemiddelde B Th-index laag, ofschoon de gemiddelde dikte groot is. De bonenopbrengst bevat volgens de aantekeningen, brede, minder brede en smalle peulen. Een enkele peul is zwart, door het weer aangetast; enkele bonen zijn donker en vlekkig. Sommige bonen vertonen de bizonderheden van vorm, — een kuiltje en een vlekje aan de uiteinden —, van de bonen van de I-lijn. De variabiliteit is groot (l = 15.37 ± 0,2, $\sigma$ = 1.05 ± 0.1). Onder de bonenopbrengsten van de I-lijn van 1936 is er geen, van welke de gemiddelden geheel overeenkomen met die van pl. 309; de gemiddelde dikte is te groot, de gemiddelde L B-index te hoog, (vgl. tab. 1 en tab. 4a). Onder de individuele bonen van pl. 309 zijn er 2 met een zeer grote breedte (b = 10.3 mm) en een hoge L B-index (L B = 72), zoals we ze niet bij vergelijkbonen van de I-lijn van 1936 aantreffen (tab. 6). Ook de classificatie van de bonenopbrengst van pl. 309 (tab. 1a) verschilt van die van overeenkomstige bonenopbrengsten van de I-lijn (tab. 5a). Er zijn te veel bonen van cl 1a en 1b, form. L B Th. De formule van de uitgangsboon van pl. 73 voor pl. 309 is $L_1$ $L_2$ B Th in niet geheel homozygote vorm.

Van de gevallen met de form. $L_1$ $L_2$ B Th, cl 1a is de gemiddelde dikte niet zeer groot; de gemiddelde L Th- en B Th-indices zijn niet zeer hoog; we moeten daarbij rekening houden met „spurious correlation". De bonen hebben overeenkomst met die van cl 2. Een enkel geval voldoet goed aan de formule $L_1$ $L_2$ B Th; de uitgangsboon voor de bonenopbrengst heeft hier deze formule in overwegend homozygote vorm.

*Cl 1b.* De formule van de uitgangsbonen is $L_1$ $l_2$ B Th, d.w.z., alle 3 afmetingen zijn groot. 40 gevallen. De bonen met de form. L B Th vormen het grote aantal bonen na de kruisingen; ze geven aan het hele materiaal het intermediaire uiterlijk.

In 5 gevallen is de formule van de gemiddelden ook L B Th (pl. 298, 242, 243, 251 en 410). Volgens de classificatie zijn er in de bonenopbrengsten een overwegend aantal bonen in cl 1. Van één geval (pl. 298, tab. 1 en 1a) geven we een korte beschrijving.

De uitgangsboon voor pl. 298 is van pl. 70. De formule van de gemiddelden van pl. 70 is L B Th ($l_m$ = 14.5 mm, tab. 21, 1947). De uitgangsboon voor pl. 70 is van pl. 81, $F_2$-1934; de formule is L B Th. De formule van de gemiddelden van pl. 81 is L B Th ($l_m$ = 14.2 mm).

De uitgangsboon van pl. 70 voor pl. 298 heeft iets kleiner lengte dan de 1ste en de 3de, laatste, boon van de peul; de breedte en de dikte zijn iets groter. De formule van de gemiddelden van pl. 298 is L B Th. Bij de bonenopbrengst staat vermeld „mooie, gave peulen". Alle individuele bonen hebben een grote dikte; de grootste dikte is th = 7.2 mm. Volgens de classificatie (tab. 1a) zijn bijna alle bonen in cl 1. Van de 3 bonen in cl 2 heeft de dikte de grenswaarde (th = 6.5, 6.5 en 6.4 mm). We hebben hier met een mooi geval van een bonenopbrengst met de form. L B Th te doen. Ook de ascendentie wijst hierop.

In 15 gevallen is de formule van de gemiddelden der bonenopbrengsten L B th, als van cl 2. Vaak heeft hier de gemiddelde dikte de grenswaarde ($th_m$ = 6.5 mm). Een geval (pl. 361) met een kleine gemiddelde dikte ($th_m$ = 6.1 mm) kan beter tot cl 2 gerekend worden; dit geldt blijkens de classificatie voor nog 4 gevallen (pl. 258, 268, 1042 en 1044). Vier gevallen hebben zeer veel bonen in cl 1 en cl 2 en enkele in cl 4, 6 en 8. Van één van deze gevallen (pl. 194) volgt hieronder een korte beschrijving. In 6 gevallen zijn er ook vrij veel bonen in andere klassen. In cl 5 en in cl 7 wordt in al deze 15 gevallen slechts één boon aangetroffen (pl. 281, 1 p 4 b; form. 1 B Th, cl 5, de lengte heeft de grenswaarde, 1 = 13.0 mm, en pl. 294, 5 p 2 b. form. 1 b Th, cl 7; de dikte is even over de grenswaarde, th = 6.6 mm). In 21 bonenopbrengsten van de I-lijn van 1936, tezamen 555 bonen, zijn volgens de classificatie geen bonen in cl 5 en in cl 7.

Van pl. 194 (tab. 1 en 1a) is de uitgangsboon van pl. 47, $F_3$-1935. De formule van de gemiddelden van de bonenopbrengsten van pl. 47 is L B Th. De uitgangsboon voor pl. 47 is van pl. 66, $F_2$-1934; de formule is L B Th. Ook de formule van de gemiddelden van pl. 66 is L B Th.

De uitgangsboon van pl. 47 voor pl. 194 is van een peul met 8 bonen. Enkele van de overige bonen hebben een iets grotere lengte (l = 14.2 mm) en een iets grotere dikte (th = 7.0—7.4 mm). De formule van de gemiddelden van pl. 194 is L B th ($th_m$ = 6.5 mm). Er zijn „brede en smalle peulen" volgens de aantekeningen. Van een „lange en smalle" peul met 7 bonen, is de breedte van de bonen 9.2—9.5 mm en de dikte 6.4—6.9 mm. Van een „brede" peul met 2 bonen is de breedte der bonen 10 en 10.4 mm, de dikte 6.6 en 7.2 mm; van een andere is b = 10.5, 10.4 en 10.1 mm. Van een smalle peul met 5 bonen is de breedte der bonen 8.2, 8.5—8.7 mm, en de dikte 5.3—5.6 mm. (Er zijn nog 2 peulen met 7 bonen in de gemeten bonenopbrengst. We hebben hier waarschijnlijk met erfelijkheid van het aantal bonen in de peul te doen. Dus als half-ras. De uitgangsboon van pl. 47 voor pl. 194 was van een peul met 8 bonen.) Volgens de classificatie zijn er zeer veel bonen in cl 1 en een minder groot aantal in cl 2. De opbrengst is samengesteld. De uitgangsboon van pl. 47 voor pl. 194 is niet homozygoot voor de form. L B Th.

In één geval (pl. 308) is de formule van de gemiddelden van de bonenopbrengst L b Th, cl 3; we bespreken het daar. Van 6 gevallen is de formule van de gemiddelden L b th, cl 4. Eén van deze gevallen (pl. 322) is een grensgeval van de gevallen met de formule der gemiddelden L B Th, cl 1 ($th_m$ = 6.5 mm); ook volgens de classificatie. De 5 overige van deze 6 gevallen bespreken we bij cl 4. Een geval (pl. 185) met de formule 1 b Th, cl 7 der gemiddelden bespreken we daar. Er zijn 11 gevallen met de formule 1 b th, cl 8, van de gemiddelden der bonenopbrengsten. Eén van deze gevallen (pl. 213) bespreken we bij cl 7, form. 1 b Th, de overige bij cl 8, form. 1 b th.

Er zijn 6 gevallen, waar de uitgangsboon niet de form. L B Th heeft en waar de formule der gemiddelden L B Th, cl 1 is. In één van deze gevallen is de formule van de uitgangsboon van de bonenopbrengst (pl. 321) L B th, cl 2 (tab. 1 en 1a). Volgens de classificatie zijn er zeer veel bonen in cl 1 en enige in cl 2. Ze heeft overeenkomst met die van pl. 322 en 323, waarvan de uitgangsbonen van dezelfde plant (pl. 78, $F_3$-1935) zijn.

De formule van de gemiddelden van pl. 78 is L B th ($th_m$ = 6.5 mm). De formule van de uitgangsboon voor pl. 78 is van pl. 81, $F_2$-1934; de formule is $L_1$ $L_2$ B th (l = 16.1 mm, th = 6.3 mm). De formule van de gemiddelden van pl. 81 is L B Th ($th_m$ = 7.0 mm). De uitgangsboon van pl. 78 voor pl. 321 heeft een veel groter lengte dan de 3 overige bonen van de peul en een kleiner dikte (l = 15.6 contra 13.7—14.3 mm; d = 6.0 contra 6.4—7.0 mm). In de bonenopbrengst van pl. 321 zijn enkele bonen met een kleine dikte (th = 6.2 mm) en overeenkomende met bonen van de I-lijn. De uitgangsboon van pl. 78 voor pl. 321, met de form. L B th is volgens de classificatie van de bonenopbrengst van pl. 321 en op grond van de gegevens der ascendentie heterozygoot en stemt zeer overeen met de form. L B Th.

Eveneens in één geval is de formule van de uitgangsboon L b Th, cl 3, en die van de gemiddelden van de bonenopbrengst (pl. 228) L B Th, cl 1.

De lengte van de uitgangsboon is zeer groot (l = 15.9 mm); de formule is dus $L_1$ $L_2$ b Th, cl 3a. De breedte (b = 8.5 mm) heeft de grenswaarde voor de form. $L_1$ $L_2$ B Th, cl 1a. Volgens de classificatie van de bonenopbrengst van pl. 228 zijn er zeer veel bonen in cl 1 (één in cl 1a); ook enige in cl 3 en cl 4. We hebben hier bij de uitgangsboon dus wel met een erfelijk iets kleinere breedte te doen.

In twee gevallen is de formule van de uitgangsboon L b th, cl 4, terwijl die van de gemiddelden der bonenopbrengst (pl. 264 en 288) L B Th is.

Volgens de classificatie behoort de bonenopbrengst van pl. 264 tot cl 1, haár uitgangsboon van pl. 59 heeft de formule L B Th in heterozygote vorm. De classificatie van pl. 288 wijst zeer veel bonen aan in cl 1, meerdere in cl 4 en enige in andere classen. De bonenopbrengst is samengesteld en de uitgangsboon van pl. 66 bevat nevens factoren in heterozygote vorm, ook enige b b- en th th-verbindingen.

Er is ook één geval met de form. l B Th, cl 5, van de uitgangsboon, terwijl de formule van de gemiddelden van de bonenopbrengst (pl. 364) L B Th is.

De gemiddelden zijn niet groot ($l_m$ = 13.2 mm). De lengte van de uitgangsboon voor pl. 364 is weinig beneden de grenswaarde (l = 12.8 mm) voor de form. L B Th. Volgens de classificatie van pl. 364 zijn er veel bonen in cl 1 en enige over alle overige classen verspreid.

Ten slotte zijn er 2 gevallen met de form. l b th van de uitgangsboon en L B Th van de gemiddelden der bonenopbrengsten (pl. 295 en 390).

De uitgangsboon voor pl. 295, $F_4$-1936 is van pl. 69, $F_3$-1935; ze heeft zeer kleine afmetingen; die van de 1e en 2e boon van de peul hebben veel grotere afmetingen; de 4e, laatste, boon van de peul heeft iets minder grote afmetingen dan de 1e en de 2e boon. De uitgangsboon is de enige zeer kleine boon van de hele bonenopbrengst van pl. 69. De formule van de gemiddelden van pl. 69 is L B Th. De uitgangsboon voor pl. 69 is van pl. 81 $F_2$-1934 en heeft de form. L B Th met zeer grote afmetingen. De formule van de gemiddelden van pl. 81 is eveneens L B Th. Volgens de classificatie van de bonenopbrengst van pl. 295 zijn bijna alle bonen in cl 1, een enkele is in cl 3 en in cl 4. We nemen aan, dat de uitgangsboon van pl. 69 voor pl. 295 met het phaenotype l b th een niet-erfelijke variant is te midden van de 3 overige bonen van de peul, die het phaenotype L B Th en L b Th hebben. Op grond van de samenstelling van de bonenopbrengst van pl. 295 nemen we verder aan, dat de uitgangsboon het genotype L B Th heeft in een in hoge mate homozygote vorm.

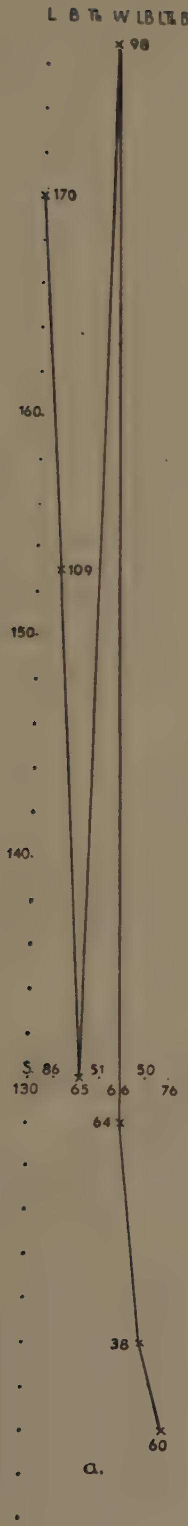
Bij de groep gevallen met de form. L B Th, cl 1 van de uitgangsbonen, vinden we een enkel geval, waar we uit de classificatie van de bonenopbrengst en de ascendentie, tot een hoge mate van homozygotie voor de form. L B Th mogen besluiten. In veel gevallen zijn er in de bonenopbrengsten, volgens de classificatie bonen in veel classen, vaak in cl 1, 2, 4 en 8, zelden in cl 5 en 7. De eerste tonen hun verwantschap met bonenopbrengsten van cl 2, met de form. L B th als van de I-lijn, de tweede met die van cl 7, form. l b Th als van de II-lijn. In bonenopbrengsten van de I-lijn zijn er volgens de classificatie, behalve een groot aantal bonen in cl 2, ook enige bonen in cl 1, 4 en 8a (tab. 5a), in die van de II-lijn zijn er behalve veel bonen in cl 7, ook enige in cl 5 en 8b (tab. 5b).

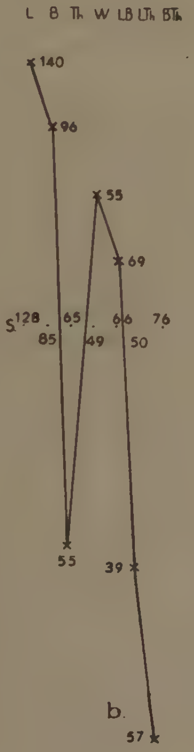
*Cl 2a.* De formule van de uitgangsboon is $L_1 L_2$ B th als van de bonen van de I-lijn met een zeer grote lengte (groter dan 15.5 mm). 6 gevallen. In 2 gevallen is de formule van de gemiddelden der bonenopbrengsten (pl. 291 en 338) $L_1 L_2$ B th, cl 2a.

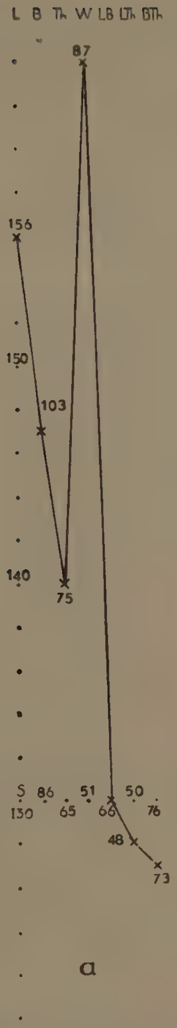
De uitgangsboon voor pl. 291 is van pl. 68, $F_3$-1935. De formule van de gemiddelden van pl. 68 is L B Th (tab. 21, 1947); de gemiddelde breedte is groot ($b_m = 9.7$ mm), de gemiddelde dikte is slechts even boven de grenswaarde ($th_m = 6.6$ mm). De uitgangsboon voor pl. 68 is van pl. 76, $F_2$-1934; de formule is L B th, cl 2 ($th = 6.4$ mm). De formule van de gemiddelden van pl. 76 is L B Th.

Van pl. 68, $F_3$-1935 zijn de uitgangsbonen genomen voor de 4 pl. 291—294, $F_4$-1936. Alle hebben een grote breedte (10.9, 10.3, 10.0, 9.6 mm). Die voor de pl. 292—294 hebben de form. L B Th; hun lengte is kleiner dan die voor pl. 291 (17.0 en 15.6, 14.3, 13.7 mm; fig. 1—4). Dergelijke bonen als de uitgangsboon van pl. 68 voor pl. 291 komen ook onder de bonen van de I-lijn van 1935 als hoge uitzondering voor (tab. 3a). Van de bonenopbrengst van pl. 291 is de gemiddelde breedte groot, de gemiddelde L B-index is hoog. De gemiddelde L B-indices van de pl. 291—294 komen zeer overeen ($LB_m = 69$ en $= 68$). Uit de characterogrammen (Fig. 1—4) zien we de kleine gemiddelde lengten en grote gemiddelde breedten en daardoor hoge gemiddelde L B-indices. Onder de bonenopbrengsten van de I-lijn van 1936 zijn er niet, waarvan de gemiddelden overeenkomen met die van de pl. 291—294 (fig. 1—4 en tab. 4): de gemiddelde L B-indices zijn veel te hoog (68 contra 62—64). De gemiddelde lengte van de bonenopbrengst van pl. 293 is kleiner dan bij bonenopbrengsten van de I-lijn van 1936 voor komt. De formule voor de uitgangsboon van pl. 68 voor pl. 293 bevat dus een kleiner aantal L-factoren dan die van de I-lijn. De gemiddelde dikte van de bonenopbrengsten van de pl. 291 en 292 ($th_m = 5.5$ mm) is kleiner dan bij bonenopbrengsten van de I-lijn van 1936 voorkomt (tab. 4). Als we hier met een genotypisch verschil te doen hebben, moeten we aannemen, dat de formule van de dikte van bonen van de I-lijn niet eenvoudig $th_1\, th_1 \ldots th_3\, th_3$ is en die van bonen van de II-lijn $Th_1\, Th_1 \ldots Th_3\, Th_3$, doch b.v., resp., $th_1\, th_1\, Th_2\, T_2\, th_3\, th_3\, th_4\, th_4$ en $Th_1\, Th_1\, th_2\, th_2\, Th_3\, Th_3\, Th_4\, Th_4$, zodat er door transgressieve variabiliteit een dikte met de formule $th_1\, th_2 \ldots th_4\, th_4$ mogelijk is, die kleiner is dan de dikte van bonen van de I-lijn.

Van de kleine bonenopbrengst van 15 bonen van pl. 291 zijn er volgens de classificatie







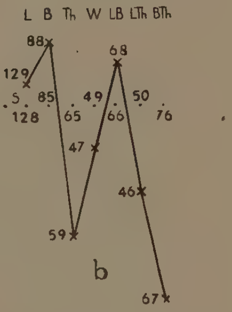


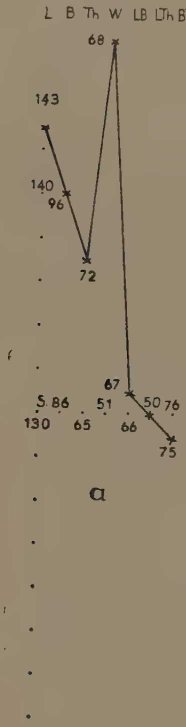
Fig. 2*a*. Characterogram of the initial bean 9 p. 3 b. of pl. 68 of $F_3$-1935 for pl. 293, $F_4$-1936.

Fig. 2*b*. Idem of the averages of pl. 293; n = 23.

Fig. 1*a*. Characterogram of the initial bean 1 p. 1 b. (1. pod, 1. bean) of pl. 68 of $F_3$-1935 for pl. 291, $F_4$-1936.

Fig. 1*b*. Idem of the mean dimensions, weight and indices of pl. 291; n = 15. S = standardcharacterogram of 1935, $S_1$ = standardcharacterogram of 1936. Dimensions in 0.1 mm, weights in cg.

14 in cl 2 en 1 in cl 4. De bonen zijn zeer overeenkomstig; alle hebben een grote breedte (b = 9.3—10.1 mm en eenmaal, van een boon, die de laatste is in de rij van de peul, is b = 8.5 mm). De L B-index is zeer hoog, varieert van 65—73 (en is eenmaal = 78). Onder de individuele bonen zijn er niet, zoals ze bij bonen van de I-lijn voorkomen. We hebben met een genotypisch verschil te doen. Het voornaamste kenmerk van de bonen is de grote breedte. Daardoor is de L B-index hoog en de B Th-index laag. De L Th-index komt overeen met die van bonen van de I-lijn. De dikte van de bonen is zeer klein (th = 5.1—6.1 en eenmaal = 4.8 mm). De lengte van de bonen is niet zeer groot, (l = 13.6—14.6 mm en eenmaal l = 13.3 en = 13.1 mm, daardoor is de L Th-index niet zeer laag, komt overeen met die van bonen van de I-lijn.



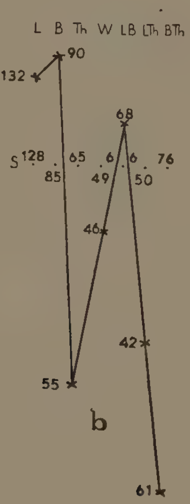
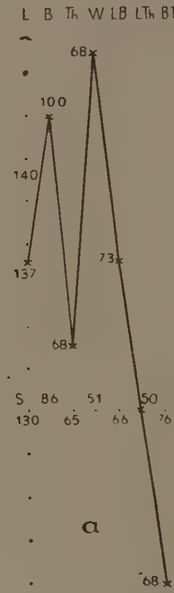


Fig. 3*a*. Characterogram of the initial bean 4 p. 4 b. of pl. 68 of $F_3$-1935 for pl. 292, $F_4$-1936.

Fig. 3*b*. Idem of the averages of pl. 292; n = 25.



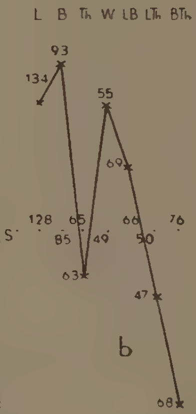


Fig. 4*a*. Characterogram of the initial bean 13 p. 2 b. of pl. 68 of $F_3$-1935 for pl. 294, $F_4$-1936.

Fig. 4*b*. Idem of the averages of pl. 294; n = 14.

We hebben bij pl. 291, voor zover het kleine aantal bonen een beoordeling toelaat, met een bonenopbrengst te doen, waarvan het genotype van de uitgangsboon van pl. 68, in hoge mate homozygoot is voor een groot aantal B-factoren en een kleiner aantal L-factoren dan het genotype van de bonen van de I-lijn bevat. Ook het aantal th-factoren in homozygote vorm is misschien groter dan dit aantal van de I-lijn.

Eveneens in 2 gevallen is de formule van de gemiddelden van de bonenopbrengsten (pl. 318 en 346) L b th, cl 4. De bonenopbrengst van pl. 318 bestaat uit slechts 4 bonen. Pl. 346 vindt beter bij cl 4 bespreking. In één geval is de formule van de gemiddelden van de bonenopbrengst (pl. 260) l B Th, cl 6 en in één (pl. 128) l b th, cl 8.

*Cl 2b.* De formule van de uitgangsboon is $L_1\ l_2$ B th als van de bonen van de I-lijn, waarvan de lengte niet groter is dan 15.5 mm, 14 gevallen. In 3 gevallen (pl. 245, pl. 282 en pl. 337) is de formule van de gemiddelden ook L B th.

Van pl. 282 komt de uitgangsboon van pl. 64, $F_3$-1935, geheel overeen met bonen van de I-lijn. de breedte is zeer groot (b = 10.0 mm; tab. 3*a*). Bonenopbrengsten met de gemiddelden als van pl. 282 treffen we bij de I-lijn van 1936 niet aan; de gemiddelde L B-index is te hoog (L B = 67; tab. 4*a*). De classificatie komt overeen met die van bonenopbrengsten van de I-lijn van 1936. Van de individuele bonen van pl. 282 is de grootste dikte, th = 7.1 mm. Bij deze boon staat aangetekend „abnormaal". Er zijn zeer veel bonen met een grote breedte en een niet grote lengte, waardoor de L B-index hoog is. De bonenopbrengst van pl. 282 komt in dit opzicht overeen met die van pl. 291 (blz. 80). Het genotype van de uitgangsboon van pl. 64 voor pl. 282 stemt overeen met dat van bonen van de I-lijn, maar bevat niet alle L-factoren.

Van pl. 337 (tab. 1 en 1a) is de uitgangsboon van pl. 87 (zie ook pl. 338, blz. 80 en pl. 336 blz. 76). De formule van de gemiddelden van pl. 87, $F_3$-1935, is L B Th ($th_m$ = 6.6 mm). De uitgangsboon is van pl. 82, $F_3$-1934, de formule is L B Th (l = 15.5 mm). De formule van de gemiddelden van pl. 82 is L b Th ($b_m$ = 8.5 mm). De uitgangsboon van pl. 87 (tab. 1) voor pl. 337 is van een peul met 7 bonen, die onderling veel verschillen. Drie bonen, waaronder de uitgangsboon hebben een kleine dikte (th = 5.6 en = 5.9 mm), drie andere hebben een grote dikte (th = 7.1—7.2 mm). Vijf bonen, waaronder de uitgangsboon, hebben een grote breedte (b = 10.0—10.4 mm). Bonenopbrengsten met gemiddelden als van pl. 337 treffen we niet bij de I-lijn van 1936 aan (tab. 4*a*). De gemiddelde L B-index is te hoog. Dezelfde te hoge L B-index hebben de pl. 336 en 338. De grootste dikte van de individuele bonen van pl. 337 is th = 6.9 mm. Onder de individuele bonen van pl. 337 zijn er geen, zoals ze, als hoge uitzondering, ook niet bij de I-lijn van 1936 voorkomen (tab. 6). Ook de classificatie (tab. 1a), treffen we zo bij bonenopbrengsten van de I-lijn van 1936 aan (tab. 1a en tab. 5a). Er zijn echter te veel bonen met een grote breedte en niet grote lengte. De uitgangsboon van pl. 87 voor pl. 337 komt zo onder de bonen van de I-lijn van 1935 als hoge uitzondering voor (tab. 1 en tab. 3*a*). De bonenopbrengst van pl. 337 lijkt op die van pl. 282 (blz. 83). De uitgangsboon van pl. 87 voor pl. 337 is in hoge mate homozygoot voor de formule van de bonen van de I-lijn, doch bezit een kleiner aantal L-factoren in homozygote vorm.

In 11 gevallen heeft de uitgangsboon de form. L B th, cl 2 en is de formule van de gemiddelden der bonenopbrengsten een andere. Eenmaal is ze L B Th, cl 1 (pl. 321, blz. 79), eenmaal L B th, cl 2 (pl. 374), eenmaal L b Th, cl 3 (pl. 121), eenmaal l b Th, cl 7 en 8 maal is de formule der gemiddelden l b th, cl 8. Deze gevallen vinden bij de resp. classen bespreking.

Er zijn in het hele materiaal 33 gevallen, waar de formule van de gemiddelden van de bonenopbrengsten L B th, cl 2 is en waar de formule van de uitgangsbonen een andere dan L B th is. In 6 van deze gevallen is de formule van de uitgangsboon $L_1\ L_2$ B Th cl 1a, in 14 gevallen is ze $L_1\ l_2$ B Th, cl 1b. Van 5 van deze gevallen komt de classificatie der bonenopbrengsten (pl. 258, 268, 361, 1042 en 1044) zeer met die van cl 2 overeen. Van pl. 361 (tab. 1 en 1a) volgt hier een korte beschrijving.

Pl. 361. De uitgangsboon is van pl. 95, $F_3$-1935. De formule van de gemiddelden van de bonenopbrengst van pl. 95 is L B Th; de gemiddelde lengte is zeer groot ($l_m = 15.5$ mm). Onder de bonenopbrengsten van de I-lijn van 1935 zijn er met nog grotere gemiddelde lengte ($l_m = 16.6$ mm). De formule van de uitgangsboon van pl. 82, $F_2$-1934 voor pl. 95 is L b th, cl. 4 ($l = 14.8$ mm); de formule van de gemiddelden van pl. 82 is L B Th. De uitgangsboon van pl. 95 voor pl. 361 is één van de 3 weinig verschillende bonen van de peul. De formule van de gemiddelden van pl. 361 is L B th, cl 2; de gemiddelde dikte is klein ($th_m = 6.1$ mm, tab. 1). Bonenopbrengsten met dergelijke gemiddelden komen bij de I-lijn voor (tab. 4*a*). Onder de individuele bonen van pl. 361 is er een enkele met een zo grote dikte, als we onder de bonen van de I-lijn van 1936 niet aantreffen. Ook volgens de classificatie (tab. 1a) zijn er te veel bonen in cl 1. Ook de uitgangsboon van pl. 95 voor pl. 361 verschilt iets van overeenkomstige vergelijkbonen van de I-lijn van 1935 (tab. 1 en tab. 3*a*). De uitgangsboon voor pl. 361 van pl. 95 is, op grond van de samenstelling van de $F_4$-bonenopbrengst van 1936 en ook volgens zijn eigen phaenotype niet geheel homozygoot voor de form. L B th van de I-lijn.

Van de overige der genoemde 33 gevallen (blz. 83) is de formule van de uitgangsboon als volgt:

In 2 gevallen is ze L b Th, cl 3. In één er van (pl. 138) zijn er volgens de classificatie zéér veel bonen in cl 2; de bonenopbrengst heeft overeenkomst met bonenopbrengsten van de I-lijn. Er is geen enkele boon in cl 3. De grote dikte van de uitgangsboon van pl. 37, $F_3$-1935 voor pl. 138 is daarom opmerkelijk. Ook de bonenopbrengst (pl. 312) van het 2e geval stemt volgens de classificatie overeen met de I-lijn. Ook hier is geen enkele boon in cl 3. De uitgangsboon met de form. L b Th, cl 3 is waarschijnlijk een niet-erfelijke variant in deze classe. In één geval is de formule van de uitgangsboon L b th, cl 4 (pl. 374). Volgens de classificatie van pl. 374 zijn er zeer veel bonen in cl 2 en in cl 4, die dus deze bonenopbrengst met de form. L B th van de gemiddelden binnen het gebied van cl 4 brengt.

Er is ook één geval met de form. l B Th, cl 5 (pl. 1045) van de uitgangsboon. De lengte heeft de grenswaarde, $l = 13.0$ mm voor de form. L B Th, cl 1. De dikte is groot. Alle bonen van de bonenopbrengst van pl. 1045 hebben een kleine dikte en behoren volgens de classificatie tot cl 2, 4 en 8a. De grote dikte van de uitgangsboon kan hierdoor niet verklaard worden.

In 4 gevallen (pl. 165, pl. 201, pl. 373 en 398) is de formule van de uitgangsboon l b Th, cl 7. De samenstelling van de bonenopbrengst van pl. 165 (tab. 1 en 1a) geeft geen aanwijzing voor de formule, die de uitgangsboon heeft. Volgens de classificatie (tab. 1a) zijn er zeer veel bonen in cl 2, cl 1 en enige in cl 4. De gemiddelde dikte heeft de grenswaarde ($th_m = 6.5$ mm) voor de form. L B Th, cl 1. De uitgangsboon voor pl. 165 van pl. 43 $F_2$-1934 is de laatste boon van een peul met 6 bonen met verschillende dikten. Bonen als de uitgangsboon treffen we in het materiaal van de II-lijn van 1935 aan (tab. 7*b*). Ook van pl. 201 ($l = 12.8$ mm) behoort de uitgangsboon, volgens de classificatie van haar bonenopbrengst, ofschoon ze een grote dikte heeft, tot het erfgebied van de I-lijn. Van pl. 373 ($l = 12.9$ mm) is de uitgangsboon van een peul met 4 bonen, die alle een grote dikte hebben. Ze komen overeen met bonen van de II-lijn, maar de lengte is iets te groot (tab. 7*b*). Volgens de classificatie van de bonenopbrengst van pl. 373 zijn er zeer veel bonen in cl 2. Dit geldt ook voor de classificatie van pl. 378. We moeten aannemen, dat de bonen van de peul, waartoe ook de uitgangsboon van pl. 98 voor pl. 373 en voor pl. 378 behoren, niet-erfelijke variaties zijn en behoren tot de erfkring van de I-lijn. Er zijn onder de bonen van de I-lijn van 1935, er ook, die lijken op bonen van de II-lijn (tab. 7*a*).

In 6 gevallen ten slotte is de formule van de uitgangsboon l b th, cl 8, terwijl de formule van de gemiddelden der bonenopbrengsten L B th is (pl. 200, 329, 347, 354, 1046 en 1047). Meestal is de lengte van de uitgangsbonen niet zeer klein ($l = 12.8$—12.1

en 11.6 mm) en de gemiddelde lengte van de bonenopbrengsten niet zeer groot (14 6, 14.3, 13.9—13.4 mm). De uitgangsboon van pl. 48, $F_3$-1935 voor pl. 200 $F_4$-1936 is van een peul met 5 bonen, die alle nog al klein zijn, terwijl de bonenopbrengst van pl. 48 overigens peulen met grotere bonen bevat. Volgens de classificatie van de bonenopbrengst van pl. 200 zijn er veel bonen in cl 2, cl 1, cl 4 en cl 8. Ofschoon de uitgangsboon van pl. 48 voor pl. 200 een relatief grote dikte heeft (th = 6.4 mm), nemen we aan, dat ze behoort tot het erfgebied van de form. L B Th en L B th (z. boven). Overeenkomstige gegevens hebben we voor de uitgangsboon van pl. 82 voor pl. 329.

Iets uitvoeriger zijn we over een ander van deze gevallen (pl. 347, tab. 1 en 1*a*, cl 4). Van pl. 347, ook van pl. 346 en 345 (blz. II 77) zijn de uitgangsbonen van pl. 91. De uitgangsboon voor pl. 347 komt zeer overeen met de 4e en de 5e, laatste, boon van de peul en er zijn meer dergelijke bonen (tab. 9). De grootste lengte van de bonen van de bonenopbrengst van pl. 347 is l = 16.3 mm. De lengte gaat dus ver uit boven die van de uitgangsboon (l = 12.8 mm). De formule van de lengte van de uitgangsboon kan dus niet l l zijn, daar de niet-erfelijke variabiliteit van de bonen van de II-lijn, dus haar variatie-breedte, voor materiaal van 1936, 7.3—13.1 mm is. Een boon met een lengte van 16.3 mm, zal in haar formule ook L-factoren in heterozygote en homozygote vorm bevatten. We nemen daarom aan, dat de lengte van de uitgangsboon in haar formule ook L L en L l-verbindingen bevat. De grootste breedte van de bonen van pl. 347 is b = 9.1 mm, die van bonen van de II-lijn is, b = 9.5 mm; de grootste dikte is th = 7.1 mm, die van bonen van de I-lijn, th = 8.2 mm; dan volgt th = 7.2 mm. De formule van de breedte en van de dikte van de uitgangsbonen van pl. 91 voor pl. 347 kan dus resp., b b en th th zijn. Volgens de classificatie (tab. 1a, cl 4) zijn er in de bonenopbrengst van pl. 347 zeer veel bonen in cl 1 (twee bonen zijn in cl 1a), ook in cl 4 en enkele in cl 2, 3 en 8a. De bonenopbrengst is samengesteld en de formule van de uitgangsboon van pl. 91 voor pl. 347 bevat heterozygoten voor de factoren van alle 3 afmetingen. De lengte heeft daarbij enige factoren als L L, de breedte en de dikte als b b en th th.

Van pl. 354 is de lengte van de uitgangsboon, die van pl. 93 is, zeer klein (l = 11.6 mm); de grootste lengte van de bonen, van pl. 354, is l = 16.1 mm. De 3 overige bonen van de peul van de uitgangsboon hebben grote afmetingen. Volgens de classificatie van pl. 354 zijn er zeer veel bonen in cl 2 en in cl 1. We nemen aan, dat de uitgangsboon van pl. 93 voor pl. 354 een niet-erfelijke variatie is uit het gebied met de form. L B Th en L B th.

Van de pl. 1046 en 1047 (tab. 1 en 1a) komen de uitgangsbonen, de gemiddelden van de bonenopbrengsten en de classificatie zeer overeen. Ze voldoen in hoge mate aan de eisen van homozygotie van de uitgangsbonen voor de form. L B th, cl 2. De formule is dus slechts l b th als phaenotype. Het kan ook zijn, dat de form. l b th hier enige factorenverbindingen als L l bevat en dat door ongeregelde dominantie de bonenopbrengst enige bonen met een grote lengte heeft; hetzelfde geldt voor de breedte. We vinden enige dominantie van de grote afmetingen over de kleine, doch ze kan ook omgekeerd zijn.

In de groep gevallen van cl 2, waar de uitgangsboon de formule L B th, cl 2, als van de I-lijn, heeft, zijn dus enkele gevallen, die geheel of bijna geheel overeenkomen met de I-lijn, waarvan dus de uitgangsboon van de bonenopbrengst de form. L B th van de I-lijn heeft in homozygote of bijna homozygote vorm. Twee gevallen komen overeen met het bij cl 2a (blz. 82) beschreven geval.

*Cl 3.* Er zijn 9 gevallen, waar de uitgangsboon van de bonenopbrengst de form. L b Th, cl 3 heeft. In één van deze gevallen (pl. 228) is de formule $L_1\ L_2$ b Th, cl 3a; de formule van de gemiddelden van de bonenopbrengst van pl. 228 is L B Th, cl 1 (zie blz. 79).

Er is één geval (pl. 212, tab. 1 en 1a), waar de uitgangsboon de form. L b Th, cl 3 heeft en de formule van de gemiddelden van de bonenopbrengst eveneens L b Th, cl 3 is. Van de uitgangsboon heeft de breedte de grenswaarde en de lengte is niet zeer groot (tab. 1); ze is dus niet een zeer kenmerkende boon van cl 3 (tab. 9). Dit geldt ook van de gemiddelden van de bonenopbrengst. Volgens de classificatie (tab. 1a) zijn er zeer veel bonen in cl 1 en enige in cl 3. Ook hier, evenals in het vorige geval (pl. 228) zijn er in de formule van de uitgangsboon enige factoren voor kleine breedte.

In 2 gevallen (pl. 138 en pl. 312) is de formule van de uitgangsboon L b Th, cl 3 en is de formule van de gemiddelden van de bonenopbrengst L B th, cl 2 (zie blz. 84). Eveneens in 2 gevallen (pl. 345 en 1037) is de formule van de uitgangsboon L b th, cl 3 en is de formule van de gemiddelden van de bonenopbrengst L b th, cl 4 (zie blz. II 77). Tenslotte is in 3 gevallen (pl. 207, 979 en 224) de formule van de uitgangsboon L b Th, cl 3 en die van de gemiddelden der bonenopbrengsten l b th, cl 8 (blz. III 79).

In 3 gevallen (pl. 308, 121 en 211), is de formule der gemiddelden van de bonenopbrengsten L b Th, cl 3, terwijl de uitgangsboon een andere formule heeft.

In één geval (pl. 308) is ze L B Th, cl 1. De gemiddelde breedte van de bonenopbrengst van pl. 308 is slechts iets kleiner dan de grenswaarde ($b_m = 8.4$ mm). Volgens de classificatie zijn er in de bonenopbrengst van 24 bonen van pl. 308, 5 bonen in cl 3, verder in alle klassen, behalve in cl 5 en 7. De bonenopbrengst is samengesteld. De uitgangsboon van pl. 172 voor pl. 308 behoort tot het erfgebied van cl 1 en cl 2 en heeft enige breedte-factoren als b b.

Van pl. 121 is de formule van de uitgangsboon L B th cl 2. Ook hier is de gemiddelde breedte weinig kleiner dan de grenswaarde ($b_m = 8.4$ mm). Volgens de classificatie van pl. 121 zijn er veel bonen in cl 1, meerdere (5 van 25) in cl 3 en enige in andere classen, ook in cl 7, geen in cl 2. Volgens de gegevens van de bonenopbrengst en van de ascendentie bevat de formule van de uitgangsboon breedte-factoren als b b en dikte-factoren als Th Th.

Van één geval ten slotte (pl. 211) is de formule van de uitgangsboon, l b Th, cl 7. De gemiddelden van de bonenopbrengst komen zeer overeen met die van pl. 212 (blz. 85), tab. 1 en 1a. Ook de indices van de uitgangsbonen van pl. 49 voor pl. 211 en 212 komen zeer overeen. Volgens de classificatie zijn er in de bonenopbrengst van pl. 211, slechts 2 bonen in cl 3 (4 in die van pl. 212) en veel bonen in cl 1 en in cl 7. Ook van pl. 211 is de bonenopbrengst samengesteld. Naast breedte-factoren als b b zijn er hier vooral ook dikte-factoren als Th Th in de formule van de uitgangsboon.

We vinden in de groep gevallen van cl 3 enige aanwijzing voor de erfelijkheid van kleine breedte. We vinden ook, dat de lengte van de bonen van cl 3 niet zeer groot en de breedte niet zeer klein is. Hier komen ook de positieve correlatie van de lengte en de breedte door de niet-erfelijke variabiliteit en de erfelijkheid door polymere factoren tot uitdrukking.

**Zoology.** — *Note on the Species of Palaemonetes (Crustacea Decapoda) found in the United States of America.* By L. B. HOLTHUIS. (Communicated by Prof. H. BOSCHMA.)

(Communicated at the meeting of December 18, 1948.)

During a year's stay at the U.S. National Museum, Washington, D.C., the American Palaemonid material of this institute as well as that of the Allan Hancock Foundation at Los Angeles, Calif. was studied for a revision of the American species of this family. As it probably will be some time before this revision is published, it was thought advisable to give a preliminary account of the species of the genus *Palaemonetes* occurring in the U.S.A. Up till now namely a large confusion existed in the conception of the taxonomic status of the various American species of this genus, while these species are very common in fresh, brackish, and salt waters of the U.S.A. and often are studied for their life history or mentioned in connection with faunistic or ecological studies.

Of all species dealt with here a short diagnosis is given, accompanied by some figures and an indication of the range of distribution. A key to the forms precedes the treatment of the separate species. More extensive descriptions and more detailed information as to the occurrence will be given in the forthcoming revision of the American Palaemonidae.

The first species of *Palaemonetes* described from the U.S.A. is *Palaemon vulgaris* of SAY (1818). SAY's description leaves no doubt whatsoever as to the identity of his material. In 1850 GIBBES described a species of prawn from freshwater of S. Carolina under the name *Hippolyte paludosa,* which species was recognized as late as 1878 by KINGSLEY to be a *Palaemonetes,* identical with *Palaemonetes exilipes,* a species described as new by STIMPSON (1871) also from fresh water of S. Carolina. In the same publication in which he described *P. exilipes,* STIMPSON gave a description of another new species of *Palaemonetes,* namely *P. carolinus.* An aberrant form, *Palaemonetes antrorum,* was found in an artesian well in Texas and described by BENEDICT (1896) as new. In 1902 RATHBUN described *Palaemonetes kadiakensis* from Alaska, and in 1921 *Palaemonetes hiltoni* was described by SCHMITT.

When KEMP (1925, Rec. Indian Mus., vol. 27, p. 315) gave his key to all species of the genus *Palaemonetes* he included the following species from the U.S.A.: *P. exilipes* (KEMP did not use the older name *paludosus*), from which he was not able to separate *P. kadiakensis, P. carolinus,* between which species and *P. hiltoni* he could not find good differences, *P. vulgaris,* and *P. antrorum.*

The study of the large material of the various species of the genus represented in the collection of the U.S. National Museum resulted in the following changes in KEMP's arrangement:

1. *Palaemonetes antrorum* is found to be so strongly different from the other species, that a separate subgenus is created for it.

2. *Palaemonetes paludosus* (the older name of GIBBES has to be used instead of the name *exilipes* of STIMPSON) was found to be restricted to that part of the U.S.A. which lies east of the Alleghenies. West of this mountain range it is replaced by a species, which shows to be identical with *Palaemonetes kadiakensis* Rathbun; it even becomes very improbable that the type specimens of RATHBUN's species really originate from Alaska, they probably are incorrectly labelled.

3. It was found that *Palaemonetes carolinus* Stimpson is based on specimens of *Palaemonetes vulgaris*, so that the former name becomes a synonym of the latter and has to disappear.

4. The form named by KEMP *Palaemonetes carolinus* shows to consist of two good species. As no name is available for either of these species, they are given new names here.

5. *Palaemonetes hiltoni* is a good species, different from *P. vulgaris*.

At present thus 7 species of *Palaemonetes* are known from the U.S.A. These species may be distinguished as follows:

1. Eyes without pigment, cornea degenerated. First and second pairs of pereiopods not very different in shape and size. Exopod of uropod with outer margin ending in a single immovable tooth. Subterranean fresh waters of Texas. Subgenus *Alaocaris*. Only species: . . . . . . . . . . . . . . . . *antrorum*

– Eyes with dark pigment, cornea well developed, globular. Second pereiopods much stronger and longer than first pair. Exopod of uropod with outer margin ending in a tooth, which at its inner side is provided with a movable spine. Fresh, brackish or salt surface waters, only accidentally subterranean. Subgenus *Palaemonetes* s.s. . 2.

2. Fused part of the two rami of upper antennular flagellum distinctly longer than free part. Carpus of second legs longer than chela. Fresh water . . . . . 3.

– Fused part of the two rami of upper antennular flagellum shorter than free part. Carpus of second legs shorter than chela. Brackish or salt water . . . . 4.

3. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove. Posterior pair of dorsal spines of telson placed midway between anterior pair and posterior margin of telson. East of Alleghenies. . . . . . *paludosus*

– Branchiostegal spine distinctly removed from anterior margin of carapace and situated some distance below branchiostegal groove. Posterior pair of dorsal spines of telson placed very close near posterior margin, and more close to that margin than to anterior pair of spines, often lying in one row with spines of posterior margin. Central U.S.A. west of Alleghenies, southern part of Central Canada, N.E. Mexico. . *kadiakensis*

4. Anterior margin of basal segment of antennula strongly produced forwards and far overreaching anterolateral spine of basal segment. S. California, N.W. Mexico . . . . . . . . . . . . . . . . . . . . . *hiltoni*

– Anterior margin of basal segment of antennula, though being convex, not overreaching anterolateral spine of basal segment. Atlantic coast of U.S.A. . . . . . 5.

5. Carpus of second leg in adult female shorter than palm, in males carpus being only very slightly (1.1 times) longer or shorter than palm. Dactylus of second leg with 2, fixed finger with 1 tooth on cutting edge. Rostrum with first two teeth of dorsal margin behind orbit. Dorsal rostral teeth reaching up to apex. Lower margin of rostrum with 3 to 5 teeth . . . . . . . . . . . . . . . . . . *vulgaris*

– Carpus of second leg in adult female much longer than palm (1.3 to 1.5 times), in males carpus almost as long as whole chela. Dactylus of second leg with one or without teeth, fixed finger without teeth on cutting edge. Rostrum with only one dorsal tooth behind orbit. . . . . . . . . . . . . . . . . . . . 6.

6. Dorsal teeth of rostrum reaching up to apex, which often is bifid. Lower margin of rostrum with 4 or 5, seldom 3 teeth. Dactylus of second leg with one distinct tooth on cutting edge, no teeth on cutting edge of fixed finger. . . . . *intermedius*

– Dorsal and ventral margins of rostrum with an unarmed stretch before tip; the latter thereby dagger shaped. Lower margin of rostrum with 2 to 4, generally 3 teeth. Dactylus as well as fixed finger of second leg without teeth on cutting edge. *pugio*

## **Alaocaris** new subgenus

Diagnosis: The rostrum is compressed, serrate on the upper margin. The carapace bears an antennal and a branchiostegal spine, a branchiostegal groove is present. The telson bears two dorsal and two posterior pairs of spines, between the latter two feathered setae are present.

The eyes are strongly degenerated and have no pigment.

The mandible bears no palp, also the other mouthparts and the branchial formula are like in *Palaemonetes* s.s.

The first and second pereiopods are very similar in shape and size. The last three legs have a shape similar to those of *Palaemonetes* s.s.

The pleopods, except the first pair, each are provided with an appendix interna, while in the male moreover an appendix masculina is present in the second pair. The uropods differ from those of *Palaemonetes* s.s. by missing the movable spine at the inner side of the final tooth of the external margin of the exopod.

Type is the only species contained at present in the subgenus, *Palaemonetes antrorum* Benedict.

### **Palaemonetes (Alaocaris) antrorum** Benedict (fig. 1*a*—*e*)

*Palaemonetes antrorum* Benedict, 1896, Proc. U.S. Nat. Mus., vol. 18, p. 615.

Diagnosis: Rostrum (fig. 1*a*) reaching about to base of last segment of antennular peduncle, apex pointed. Rostral formula: $\frac{3)\ 8-12}{0}$. Branchiostegal spine removed from anterior margin of carapace.

Telson (fig. 1*b*) with anterior pair of dorsal spines placed slightly behind the middle, posterior pair situated close near posterior margin of telson.

Eyes bullet shaped, cornea degenerated, without pigment.

Antennular peduncle (fig. 1*c*) with rounded anterior margin of basal segment reaching about as far forwards as anterolateral spine. Upper antennular flagellum with free part of shorter ramus much longer than fused part.

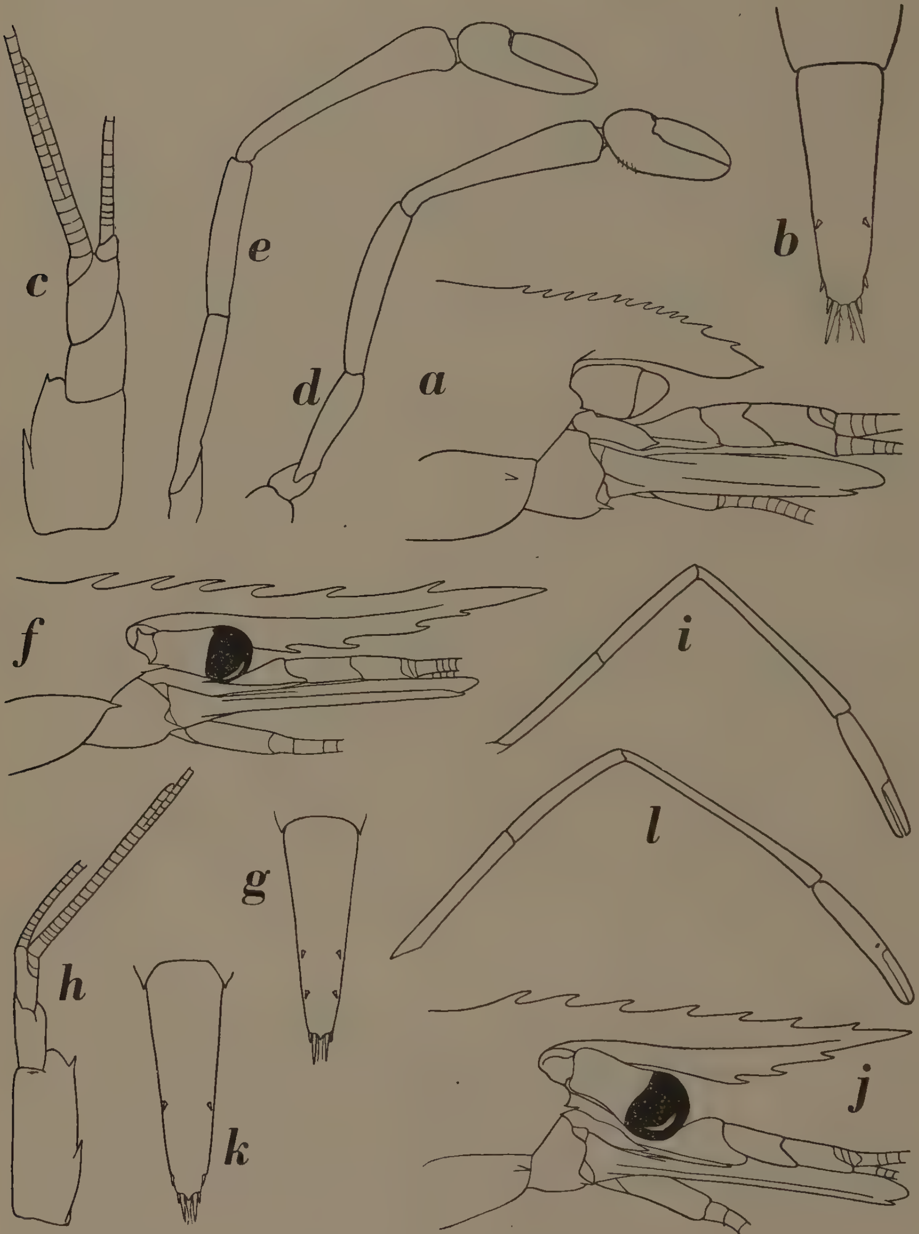


Fig. 1. *a—e, Palaemonetes* (*Alaocaris*) *antrorum* Benedict, paratype. *a,* anterior part of body in lateral view; *b,* telson in dorsal view; *c,* antennula; *d,* first pereiopod; *e,* second pereiopod. *f—i, Palaemonetes* ( *Palaemonetes*) *paludosus* (Gibbes). *f,* anterior part of body in lateral view; *g,* telson in dorsal view; *h,* antennula; *i,* second pereiopod. *j—l, Palaemonetes* (*Palaemonetes*) *kadiakensis* Rathbun. *j,* anterior part of body in lateral view; *k,* telson in dorsal view; *l,* second pereiopod. *a—e,* × 15; *f—l,* × 6.

Fingers of first pereiopod (fig. 1*d*) about twice as long as palm. Second leg (fig. 1*e*) differing only slightly from first, being somewhat longer. Carpus half again as long as chela.

Uropods with outer margin of exopod ending in a tooth, which bears no movable spinule at its inner side.

Size: Up to 18 mm.

Distribution: The species is known only from subterranean waters near San Marcos, Texas, where it occurs in considerable numbers.

*Palaemonetes antrorum* differs from all other species by the absence of pigment in the eyes, and by the degenerate cornea. The blind prawns from Cuban caves, described in literature as *Palaemonetes eigenmanni* Hay, *P. calcis* Rathbun, *P. inermis* Chace and *P. gibarensis* Chace on examination proved to be no *Palaemonetes* at all as no branchiostegal spine is present. A new genus **Troglocubanus** is erected for them (type species: *Palaemonetes eigenmanni* Hay, 1903, Proc. U.S. Nat. Mus., vol. 26, p. 431, fig. 2). This genus is characterized by the compressed rostrum, the absence of hepatic and branchiostegal spines, by the unpigmented eyes, which have the cornea degenerated, and by the absence of a mandibular palp.

### Subgenus **Palaemonetes** Heller, 1869 s.s.

#### **Palaemonetes (Palaemonetes) paludosus** (Gibbes) (fig. 1*f*—*i*)

*Hippolyte paludosa* Gibbes, 1850, Proc. Amer. Ass. Adv. Sci., vol. 3, p. 197.
*Palaemonetes exilipes* Stimpson, 1871, Ann. Lyc. nat. Hist. New York, vol. 10, p. 130.

Diagnosis: Rostrum (fig. 1*f*) reaching about to the end of the scaphocerite. Rostral formula: $\frac{1\text{—}2)\ 5\text{—}9}{1\text{—}4}$, generally $\frac{1)\ 6\text{—}8}{3\text{—}4}$. Branchiostegal spine situated on anterior margin of carapace just below branchiostegal groove; this spine sometimes slightly removed from anterior margin, but always reaching beyond the margin with larger part of its length.

Telson (fig. 1*g*) with anterior dorsal pair of spines placed slightly behind middle, posterior pair about midway between anterior pair and posterior margin of telson.

Eyes with cornea well developed, globular and well pigmented.

Basal segment of antennular peduncle (fig. 1*h*) with rounded anterior margin reaching slightly beyond anterolateral spine. Upper antennular flagellum with free part of shorter ramus about $^1/_3$ of length of fused part.

First legs with fingers about as long as palm. Second legs (fig. 1*i*) much stronger and longer than first. Fingers $^3/_4$ of length of palm, cutting edges without teeth. Carpus about 1.5 times as long as chela.

Exopod of uropod with outer margin ending in an immovable tooth, which bears at its inner side a movable spine.

Size: Up to 46 mm.

Distribution: Fresh waters east of the Alleghenies, from New Jersey to Florida. Occasionally in localities west of the Alleghenies (Mississippi, Louisiana, Oklahoma, Texas), possibly introduced there.

## **Palaemonetes (Palaemonetes) kadiakensis** Rathbun (fig. 1*j*—*l*)

*Palaemonetes kadiakensis* Rathbun, 1902, Proc. U.S. Nat. Mus., vol. 24, p. 93.

Diagnosis: Rostrum (fig. 1*j*) straight, reaching about end of scaphocerite. Rostral formula: $\frac{1)\ 5-10}{0-4}$, generally $\frac{1)\ 7}{2}$. Branchiostegal spine placed some distance behind anterior margin of carapace, at most reaching with the tip beyond the margin. Branchiostegal groove touching anterior margin of carapace distinctly dorsally of branchiostegal spine.

Telson (fig. 1*k*) with anterior pair of dorsal spines behind middle of telson, posterior pair placed very close to posterior margin of telson, lying far closer to this margin than to anterior pair of spines.

Eyes as in *P. paludosus*. Antennula with anterior margin of basal segment sometimes much produced anteriorly. Upper flagellum with fused part more than thrice as long as free part of shorter ramus.

Pereiopods and uropods as in previous species.

Size: Up to 53 mm.

Distribution: Fresh waters of Central U.S.A. west of the Alleghenies from the Great Lakes to the Gulf coast, also in S. Ontario (Canada) and N.E. Mexico.

## **Palaemonetes (Palaemonetes) hiltoni** Schmitt (fig. 2*a*—*d*)

*Palaemonetes hiltoni* Schmitt, 1921, Univ. Calif. Publ. Zool., vol. 23, p. 36, pl. 12 fig. 5.

Diagnosis: Rostrum (fig. 2*a*) reaching end of scaphocerite. Rostral formula $\frac{1)\ 8-11}{2-3}$. Dorsal teeth of rostrum generally regularly divided over upper margin. Branchiostegal spine on anterior margin of carapace, just below branchiostegal groove.

Telson with anterior pair of dorsal spines situated in its middle, posterior pair about halfway between anterior pair and posterior margin of telson.

Eyes like in *P. paludosus*.

Antennular peduncle (fig. 2*b*) with anterior margin of basal segment strongly produced anteriorly and distinctly overreaching the anterolateral spine. Free part of shorter ramus of upper antennular flagellum about 1.5 times as long as fused part.

First pereiopod with fingers as long as palm. Second leg (fig. 2*c*) much stronger than first. Dactylus (fig. 2*d*) in ovigerous females with two teeth, fixed finger with one tooth on cutting edge. Palm 1.3 to 1.4 times as long as fingers, carpus as long as palm and half the length of the fingers.

Uropods as in *P. paludosus*.

Size: Up to 24 mm long.

Distribution: S. California and N.W. Mexico (Sonora and Sinaloa States). Living probably in brackish water.

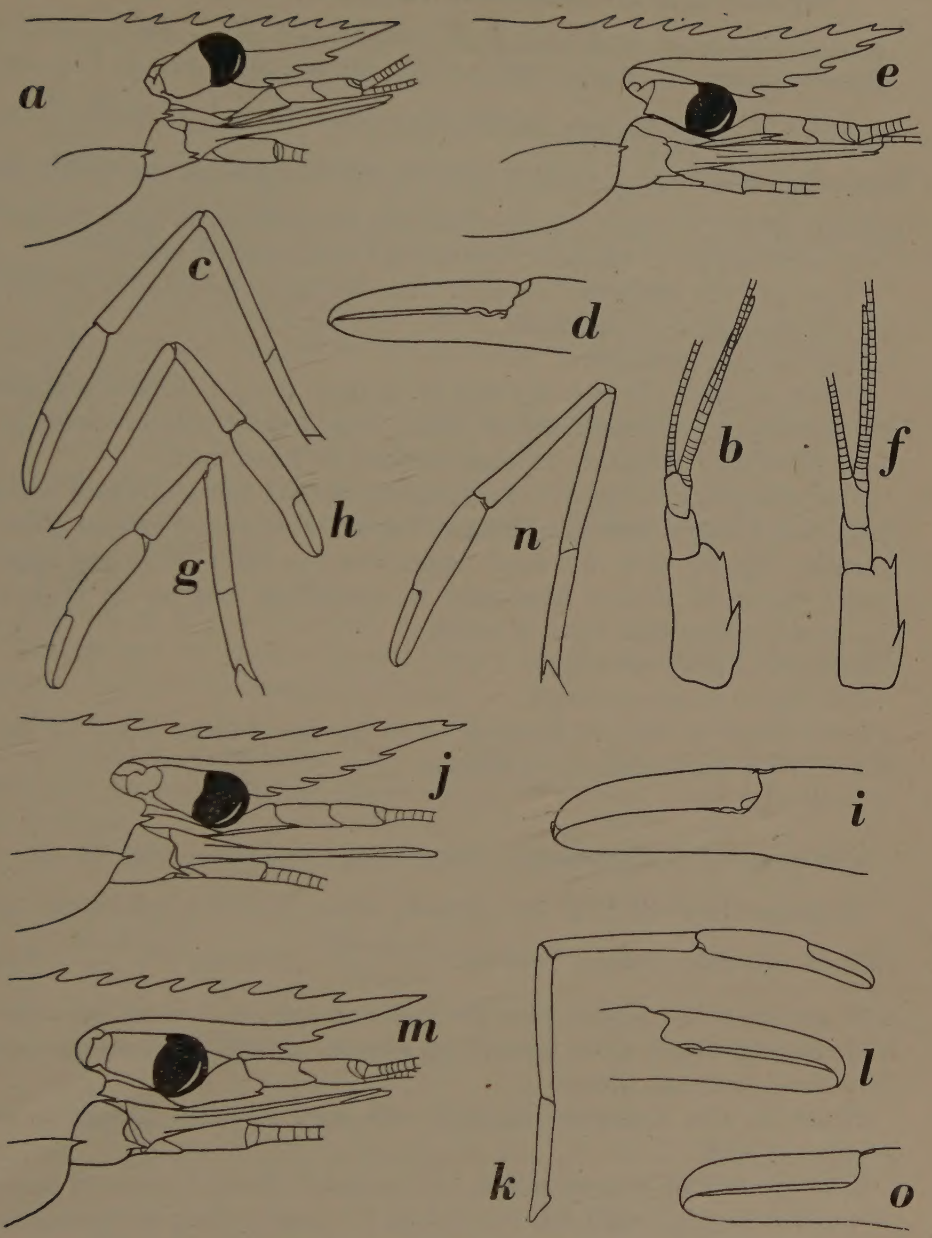


Fig. 2. *a—d, Palaemonetes (Palaemonetes) hiltoni* Schmitt. *a,* anterior part of body in lateral view; *b,* antennula; *c,* second pereiopod of female; *d,* fingers of second pereiopod. *e—i, Palaemonetes (Palaemonetes) vulgaris* (Say). *e,* anterior part of body in lateral view; *f,* antennula; *g,* second pereiopod of female; *h,* second pereiopod of male; *i,* fingers of second pereiopod of female. *j—l, Palaemonetes (Palaemonetes) intermedius* new species. *j,* anterior part of body in lateral view; *k,* second pereiopod of female; *l,* fingers of second pereiopod of female. *m—o, Palaemonetes (Palaemonetes) pugio* new species. *m,* anterior part of body in lateral view; *n,* second pereiopod of female; *o,* fingers of second pereiopod of female. *a—c, e—h, j, k, m, n,* × 6; *d, i, l, o,* × 15.

### **Palaemonetes (Palaemonetes) vulgaris** (Say) (fig. 2*e*—*i*)

*Palaemon vulgaris* Say, 1818, Journ. Acad. nat. Sci. Phila., vol. 2, p. 248.
*Palaemonetes carolinus* Stimpson, 1871, Ann. Lyc. nat. Hist. New York, vol. 10, p. 129.

Diagnosis: Rostrum (fig. 2*e*) reaching about to end of scaphocerite. Rostral formula: $\frac{2)\ 8\text{—}11}{3\text{—}5}$. Teeth placed regularly over upper margin of rostrum, tip never dagger shaped, often bifid. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Pleura of fifth abdominal segment with the tip rectangular or slightly acute. Telson as in *P. paludosus*.

Eyes as in *P. paludosus*.

Antennular peduncle (fig. 2*f*) with the anterior margin of basal segment not reaching beyond anterolateral spine. Free part of shorter ramus of upper antennular flagellum 1.5 times as long as fused part.

First pereiopod with fingers as long as palm. Second legs (fig. 2*g*, *h*) much stronger and longer than first. Fingers 0.6 to 0.75 times as long as palm. Dactylus (fig. 2*i*) with two, fixed finger with one tooth on cutting edge. Carpus distinctly shorter than palm in female, as long as or slightly (1.1 times) longer than palm in males.

Uropods as in *P. paludosus*.

Size: Up to 42 mm in length.

Distribution: In salt or brackish water of the Atlantic coast of the U.S.A., from Massachusetts to Texas.

### **Palaemonetes (Palaemonetes) intermedius** new species (fig. 2*j*—*l*)

Diagnosis: Rostrum (fig. 2*j*) reaching about end of scaphocerite, tip directed upwards. Rostral formula: $\frac{1)\ 7\text{—}10}{3\text{—}5}$, generally $\frac{1)\ 8\text{—}9}{4\text{—}5}$. The teeth are regularly divided over the upper margin of rostrum, tip often bifid. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Pleura of fifth abdominal segment with apex rounded. Telson as in previous species.

Eyes and antennular peduncle as in *P. vulgaris*. Shorter ramus of upper antennular flagellum with free part 1.2 to 1.7 times as long as fused part.

First leg with fingers as long as palm. Second leg (fig. 2*k*) longer and stronger than first. Fingers (fig. 2*l*) 0.6 to 0.8 times as long as palm. Dactylus with one tooth on cutting edge, cutting edge of fixed finger unarmed. Carpus 1.2 to 1.5 times as long as palm.

Uropods as in *P. vulgaris*.

Size: Up to 37 mm in length.

Distribution: Brackish water of the Atlantic coast of the U.S.A. from Massachusetts to Texas.

### **Palaemonetes (Palaemonetes) pugio** new species (fig. 2*m*—*o*)

Diagnosis: Rostrum (fig. 2*m*) reaching about to the end of scaphocerite, straight, with the top sometimes curved upwards. Rostral formula: $\frac{1)\ 7\text{—}10}{2\text{—}4}$, generally $\frac{1)\ 8\text{—}9}{3}$. Distal part of both upper and lower margin unarmed, tip thereby becoming dagger shaped. Branchiostegal spine situated on anterior margin of carapace, just below branchiostegal groove.

Fifth abdominal segment with the pleura ending in a small acute tooth, which sometimes is very small. Telson as in *P. vulgaris.*

Eyes and antennular peduncle as in *P. vulgaris.* Upper antennular flagellum with the free part of shorter ramus 1.1 times to twice as long as the fused part.

First leg with fingers as long as palm. Second legs (fig. 2*n*) stronger and longer than first. Fingers (fig. 2*o*) 0.6 to 0.8 times as long as palm, no teeth present on the cutting edges of dactylus and fixed finger. Carpus 1.3 to 1.5 times as long as palm in females, in males carpus being about as long as whole chela.

Uropods as in *P. vulgaris.*

Size: Up to 50 mm in length.

Distribution: Brackish to almost fresh water of the Atlantic coast of the U.S.A. from Massachusetts to Texas.

*Palaemonetes vulgaris, P. intermedius* and *P. pugio* have the same range of geographic distribution, but probably their ecological habitats are different. *P. pugio* at least prefers water of a much lower salinity than *P. vulgaris.*